On log-normal convolutions: An analytical-numerical method with applications to economic capital determination

Edward Furman*, Dan Hackmann†, Alexey Kuznetsov‡

October 10, 2019

Abstract

We put forward an efficient algorithm for approximating the sums of independent and log-normally distributed random variables. Namely, by combining tools from probability theory and numerical analysis, we are able to compute the cumulative distribution functions of the just-mentioned sums to a high precision and in a relatively short computing time. We illustrate the effectiveness of the new method in the contexts of the individual and collective risk models, aggregate economic capital determination, and economic capital allocation.

Keywords: log-normal distribution, convolution, generalized gamma convolution, Padé approximation, individual risk model, collective risk model, economic capital

JEL Classification: C02, C46, C63

1 Introduction

The log-normal distribution has been found appropriate for modelling losses originating from a great variety of non-life insurance risks (e.g., Mikosch, 2009; Klugman et al., 2012). More specifically, Kleiber and Kotz (2003) mention applications in property, fire, hurricane, and motor insurances, to name just a few (also, e.g., Dropkin, 1964; Bickerstaff, 1972; O'Neill and Wells, 1972). Furthermore the standard

*Dept. of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON, M3J 1P3, Canada. Email: efurman@mathstat.yorku.ca

†Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University, Linz, Austria. Email: dan@danhackmann.com

‡Dept. of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON, M3J 1P3, Canada. Email: kuznetsov@mathstat.yorku.ca

formula of the European Insurance and Occupational Pensions Authority explicitely assumes the log-normality of insurers' losses (EIOPA-14-322, 2014). Finally, the role of the log-normal distribution is at least as profound in finance, where it serves as the canonical model describing stock price returns (e.g., Sprenkle, 1964; Milevsky and Posner, 1998). For a multitude of applications of the log-normal distribution in other areas, we refer to Limpert et al. (2001) and references therein.

Recall that the random variable (r.v.) L is said to be distributed log-normally with parameters $\mu \in (-\infty, \infty)$ and $\sigma > 0$, succinctly $L \sim LN(\mu, \sigma^2)$, if its probability density function (p.d.f.) is given by

$$f_L(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right) \quad \text{for } x > 0. \tag{1}$$

Although L is merely an exponential of the well-understood Gaussian r.v., it has been puzzling researchers for decades. For instance, an explicit expression for the Laplace transform of (1) has eluded mathematicians thus far, and the existing series representations are rather cumbersome (Leipnik, 1991). Recently, Asmussen et al. (2016a) reported a closed form approximation for the Laplace transform of (1) that works quite well for small values of the σ parameter and is asymptotically equivalent to the approximated Laplace transform.

In this paper we are interested in a related problem of *log-normal convolutions*. In fact, the convolutions of log-normal distributions have drawn considerable attention of researchers and practitioners due to the fundamental importance of log-normal sums in engineering, biology, ecology, and economics, as well as in actuarial science and finance. It is not surprising therefore that the existing contributions on the topic are abundant and span all of the just-mentioned areas (see Dufresne, 2008; Asmussen et al., 2011, for recent literature reviews).

The problem is admittedly very intricate, and no explicit solution is generally available. The existing methods to approximate the log-normal convolutions can be associated with the following three main threads, or their variants: (i) moment matching method and its modifications, (ii) series representations of the Laplace transform, and (iii) asymptotic results.

The moment matching method is akin to the idea of approximating the convolutions of log-normal distributions by means of other distributions. It seems that for the first time this method was documented in Fenton (1960), who employs the log-normal distribution with the first and second moments being equal to the first and second moments of the desired log-normal sum (e.g., Beaulieu et al., 1995; Milevsky and Posner, 1998). The idea is further refined in Chen et al. (2008) and Zhang and Song (2008), who utilize the four-parameter Pearson IV family of distributions to approximate the distribution of log-normal sums. An alternative - analytical - way to approximate the convolutions of log-normal distributions is by inverting the corresponding Laplace transform, which is to this end expanded into a series. In this respect, Holgate (1989) discusses several techniques for approximating the characteristic function of the log-normal distribution using the re-summation of divergent series, and also provides asymptotic

approximations, and Leipnik (1991) derives a convergent series representation for the characteristic function (though Miles (2018) has recently shown that Leipnik's integral representation and the resulting series expansion are incorrect). Last but not least, asymptotic approximations of the convolutions of log-normal distributions are derived in Asmussen and Rojas-Nandayapa (2008) (see also Dhaene et al., 2008; Gulisashvili and Tankov, 2016). Methods that do not immediately fit within the research directions above also exist (e.g., Vanduffel et al., 2008; Asmussen et al., 2016b; Botev et al., 2018; Asmussen, 2018).

Unfortunately, none of the existing approaches is perfect and some of them may deliver inaccurate results for certain values of parameters. This is particularly so for large values of the σ parameter, and when a small number of log-normally distributed summands are considered. The reasons are that in the former case the distribution of $L \sim LN(\mu, \sigma^2)$ would have heavier tails, and in the latter case the Central Limit Theorem would not apply (Asmussen et al., 2011). The solution that we propose in this paper is distinct in that it is based on a synthesis of tools from probability theory, complex analysis and numerical analysis. Our main contribution is that we construct an approximant r.v., $\tilde{L}$, such that the Laplace transform of its distribution converges uniformly and exponentially fast to the Laplace transform of the distribution of $L \sim LN(\mu, \sigma^2)$. The approximating Laplace transform of the desired convolution then follows immediately, and the cumulative distribution function (c.d.f.) of the convolution is obtained via routine Laplace transform inversion techniques. The proposed method performs very well in the body of the distribution of the sum and surprisingly well in the intermediate tail region, it is quick when tackling numerous summands of varying tail thickness, allows for any level of accuracy, and, last but not least, it is suitable for computing the c.d.f.'s of the sums of independent and not-necessarily log-normally distributed r.v.'s.

We illustrate the efficiency of our approach with a few examples borrowed from the context of economic capital determination and allocation within the individual and collective risk models. We recall that, for mutually independent r.v.'s $X_1, X_2, \ldots$, the r.v.

$$S_n := X_1 + \cdots + X_n$$

is called an Individual Risk Model (IRM). If the r.v.'s $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.), and N is a random variable that takes values in $\{0, 1, 2, \ldots\}$ and independent on $\{X_i\}_{i \geq 1}$, then we call $S_N := X_1 + \cdots + X_N$ a Collective Risk Model (CRM). In the following we often write S_N irrespective of whether the CRM or IRM is considered. This is admittedly a slight abuse of notation, but it simplifies the exposition greatly.

The aforementioned choice of applications is obviously not ad hoc. Indeed, besides the clear link to the notion of convolutions, the individual and, also, collective risk models have been taught to actuarial students for many years now, and the two models have manifested ubiquitously in both theoretical and practical loss modelling (e.g., Kaas et al., 2008; Klugman et al., 2012). In addition, from the point of view of the modern insurance regulation (e.g., Solvency II, and equivalents), it is essential to evaluate

the economic capital required for supporting the aggregate risk r.v. S_N (herein we use the notions "risk" and "loss" interchangeably). In this paper we touch on all of the above. Namely, we assume that $\{X_i\}_{i\geq 1}$ are independent and log-normally distributed r.v.'s and compute the Value-at-Risk (VaR)

$$\mathrm{VaR}_q[S_N] := \inf\{s \in \mathbb{R} : \mathbb{P}[S_N \leq s] \geq q\} \tag{2}$$

and the Conditional Tail Expectation (CTE)

$$\mathrm{CTE}_q[S_N] := \mathbb{E}[S_N |\ S_N > \mathrm{VaR}_q[S_N]] \ \text{ for } S_N \text{ with finite mean} \tag{3}$$

risk measures, where $q \in [0,\ 1)$ is the prudence parameter, and N is deterministic or random. The VaR and the CTE risk measures do not require advertisement, as their popularity in insurance and banking is immense (Artzner et al., 1999; McNeil et al., 2005; Denuit et al., 2006). For the situations when the variability of the tail risk is of interest, we compute the Tail Variance (TV) risk measure

$$\mathrm{TV}_q[S_N] := \mathbf{Var}[S_N |\ S_N > \mathrm{VaR}_q[S_N]] \ \text{ for } S_N \text{ with finite variance} \tag{4}$$

(Furman and Landsman, 2006). Furthermore, we compute the economic capital allocations that correspond to risk measures (3) and (4) (Furman and Zitikis, 2008a; Dhaene et al., 2012, and references therein) when the r.v.'s $\{X_i\}_{i\geq 1}$ are distributed log-normally, and N is deterministic or random.

The rest of the paper is organized as follows. In Section 2 we describe our method, formulate and prove the main results. Further, in Section 3 we show how to implement our approximation scheme, and then in Sections 4, 5 and 6 elucidate it with examples. In Section 7 we discuss the computation time of our algorithm, and Section 8 provides concluding remarks. Some well-known but worthy to mention details about the numerical inversion of the Laplace transform are relegated to Appendix A.

2 The analytical basis for the proposed method

Let us first present some important notation. For a non-negative r.v. X, we denote by $\phi_X(z) := \mathbb{E}[\exp(-zX)]$, $\mathrm{Re}(z) \geq 0$ the Laplace transform of X and by $F_X(x) := \mathbb{P}(X \leq x)$ the c.d.f. of X. The inverse Laplace transform is denoted by $\mathcal{L}^{-1}$.

Now we briefly explain the main ideas behind our method. Our goal is to construct a random variable, $\widetilde{L}$, whose Laplace transform $\phi_{\widetilde{L}}(z)$ is easily computable and approximates the Laplace transform of $L \sim LN(0, \sigma^2)$. Once this is done, we are able to compute any quantity of interest by using Laplace transform techniques. Our approximation, $\widetilde{L}$, is of the form

$$\widetilde{L} = \sum_{i=1}^{m} \Gamma_i, \tag{5}$$

where $\Gamma_i \sim Ga(\alpha_i, \beta_i),\ i = 1, \ldots, m$ are independent gamma distributed r.v.'s with rate and shape parameters $\beta_i > 0$ and $\alpha_i > 0$, respectively. We show in the sequel that, for each $m \geq 1$, it is possible to choose the parameters α_i and β_i in such a way that as $m \to +\infty$ the Laplace transforms $\phi_{\widetilde{L}}(z)$ converge to $\phi_L(z)$ exponentially fast (on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$), and we present an algorithm for computing these parameters.

Without loss of generality, we restrict ourselves to considering only unit scale log-normally distributed r.v.'s $L \sim LN(0, \sigma^2)$, or, succinctly, $L \sim LN(\sigma^2)$. Once we have constructed a r.v., $\tilde{L}$, whose distribution approximates the distribution of $L \sim LN(\sigma^2)$, it can be used to approximate the Laplace transform of the more general three parameter log-normal p.d.f. (e.g., O'Neill and Wells, 1972)

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-\tau)-\mu}{\sigma}\right)^2\right) \quad \text{for } x > \tau. \tag{6}$$

This is obviously true because the Laplace transform of (6) is given by $e^{-z\tau}\phi_L\left(ze^{\mu}\right)$ for $\mathrm{Re}(z) \geq 0$, and so its approximation is simply $e^{-z\tau}\phi_{\widetilde{L}}\left(ze^{\mu}\right)$ for $\mathrm{Re}(z) \geq 0$.

Once we know how to approximate a single log-normal r.v., it is clear how to approximate a sum of such r.v.'s. For example, consider the Individual Risk Model. Let the r.v.'s $L_i \sim LN(\sigma_i^2),\ i = 1, \ldots, n$ be independent and let $S_n = L_1 + \cdots + L_n$. Then we can approximate S_n by $\widetilde{S}_n := \widetilde{L}_1 + \cdots + \widetilde{L}_n$, where $\{\widetilde{L}_i\}_{1 \leq i \leq n}$ are independent and each $\widetilde{L}_i$ is an approximation to the corresponding r.v. L_i. The Laplace transform of $\widetilde{S}_n$ can then be easily computed via

$$\phi_{\widetilde{S}_n}(z) = \prod_{i=1}^{n} \phi_{\widetilde{L}_i}(z),\ Re(z) \geq 0, \tag{7}$$

and any quantity of interest can be found by inverting the corresponding Laplace transform. For example, if we are interested in the c.d.f of $\widetilde{S}_n$, we compute it via

$$F_{\widetilde{S}_n}(x) = \mathcal{L}^{-1}\left\{\frac{\phi_{\widetilde{S}_n}(z)}{z}\right\}(x) \text{ for } x \geq 0. \tag{8}$$

In a similar fashion we can deal with the Collective Risk Model. If N is a r.v. with the probability generating function (p.g.f.) $G_N(z) := \mathbb{E}\left[z^N\right]$, and the r.v.'s $L_i,\ i = 1, \ldots, N$ are (i) distributed as a canonical r.v. $L \sim LN(\sigma^2)$ and (ii) independent from each other and from the r.v. N, then we can approximate the compound sum S_N by $\widetilde{S}_N = \widetilde{L}_1 + \cdots + \widetilde{L}_N$, where $\{L_i\}_{i \geq 1}$ are independent and distributed as $\widetilde{L}$, which is our approximation to L. The Laplace transform of $\widetilde{S}_N$ is easily computed via

$$\phi_{\widetilde{S}_N}(z) = G_N\left(\phi_{\widetilde{L}}(z)\right),\ Re(z) \geq 0. \tag{9}$$

The rest of this section presents our approach for approximating the Laplace transform of the r.v. $L \sim LN(\sigma^2)$.

2.1 Generalized Gamma Convolutions and the approximation of the log-normal Laplace transform

In view of the discussion hitherto, constructing an approximation $\widetilde{L}$ to $L \sim LN(\sigma^2)$ is fundamental for us. To this end, we turn to a class of distributions that contains the log-normal distribution as a special case. Indeed, it often happens in the mathematical sciences that generalizing an object highlights its essence and in this way helps to understand it better, and our problem is not an exception.

More formally, it is known that the distribution of $L \sim LN(\sigma^2)$ is a limiting distribution of a sequence of convolutions of gamma distributions, and it is thus infinitely divisible (Thorin, 1977; Bondesson, 2002). This motivates the introduction and study of the class of Generalized Gamma Convolutions as follows.

Definition 1 (Thorin (1977); Bondesson (1992))**.** The distribution on $[0,\ \infty)$ of the r.v. X is a Generalized Gamma Convolution (GGC) if its Laplace transform is

$$\phi_X(z) = \exp\Big(- az - \int_0^\infty \ln(1 + z/t)U(\mathrm{d}t)\Big) \text{ for } \mathrm{Re}(z) \geq 0, \tag{10}$$

where $a \in [0,\ \infty)$ is a constant, and $U(\mathrm{d}t)$ is a positive Radon measure, also called *Thorin* measure, which must satisfy

$$\int_0^\infty \min(|\ln(t)|, 1/t)U(dt) < \infty.$$

Besides the already-mentioned log-normal and gamma, the class of GGC's comprises such well-known to actuaries distributions, as Pareto, inverse Gaussian, inverse gamma, Beta prime, Weibull (with shape parameter less than one), among many others.

Remark 1. Set $a = 0$ in (10) for convenience and without loss of generality, and notice that if $U(dt)$ is a discrete measure, then (10) is the Laplace transform of a finite convolution of gamma distributions, thus motivating the name *GGC*.

We now briefly explain the main idea behind our approach. Let X be a random variable with a distribution in the class of GGC's. The distribution of X can be approximated arbitrarily well by the distributions of finite sums of the form

$$\tilde{X} := \sum_{i=1}^{m} \Gamma_i, \tag{11}$$

where $\Gamma_i \sim Ga(\alpha_i, \beta_i)$ denote independent gamma distributed r.v.'s with shape parameters $\alpha_i > 0$ and rate parameters $\beta_i > 0$. Note that approximating the distribution of X by the distribution of $\tilde{X}$ of form (11) is equivalent to approximating the Laplace transform $\phi_X(z) = \mathbb{E}[\exp(-zX)]$ by functions of the form

$$\phi_{\tilde{X}}(z) = \mathbb{E}[\exp(-z\tilde{X})] = \prod_{i=1}^{m} \mathbb{E}[\exp(-z\Gamma_i)] = \prod_{i=1}^{m} (1 + z/\beta_i)^{-\alpha_i} \text{ for } \mathrm{Re}(z) \geq 0. \tag{12}$$

Before stating the main result in this section, we need to present one definition. For $c > 0$ and a positive r.v. X we define *the Esscher transform* r.v. $X^{(c)}$ via the distribution function

$$\mathbb{P}(X^{(c)} \in \mathrm{d}x) = \frac{e^{-cx}}{\mathbb{E}[e^{-cX}]}\mathbb{P}(X \in \mathrm{d}x) \text{ for } x > 0.$$

(e.g. Asmussen et al., 2016b, for applications of the Esscher transform in the context of the log-normal distribution). Now we are ready to state our first main result in this section.

Theorem 1. *Let the r.v. X have a distribution in the class of GGC's, and assume that X is not of the form in (11). Fix $z^* > 0$ and $m \in \mathbb{N}$. There exist positive numbers $\{\alpha_i\}_{1\le i\le m}$ and $\{\beta_i\}_{1\le i\le m}$ such that the r.v. $\widetilde{X}$ defined by (11) satisfies*

$$\mathbb{E}[(\widetilde{X}^{(z^*)})^k] = \mathbb{E}[(X^{(z^*)})^k] \tag{13}$$

for $1 \le k \le 2m$. Moreover, the numbers $\{\alpha_i\}_{1\le i\le m}$ and $\{\beta_i\}_{1\le i\le m}$ are unique (up to permutation).

Definition 2. The random variable $\tilde{X}$, whose distribution is uniquely characterized by Theorem 1, is called an (m, z^*)-approximant of X.

Our second main result shows that (for any fixed z^*) as m increases to $+\infty$, the distribution of the (m, z^*)-approximant of X converges weakly to the distribution of X, and the convergence is very fast in the Laplace transform domain.

Theorem 2. *Let the r.v. X have a distribution in the class of GGC's, and assume that X is not of the form in (11). Fix $z^* > 0$ and define, for each $m \in \mathbb{N}$, the r.v. $\widetilde{X}_m$ to be the (m, z^*)-approximant of X. Then the functions $\phi_{\widetilde{X}_m}(z) := \mathbb{E}[\exp(-z\widetilde{X}_m)]$ converge to $\phi_X(z) = \mathbb{E}[\exp(-zX)]$ as $m \to +\infty$, exponentially fast and uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. In particular, $\widetilde{X}_m \xrightarrow{d} X$.*

Remark 2. Theorem 1 warrants that the proposed approximating scheme is parsimonious. More specifically, in order to match the first $2m$ moments of the Esscher transform of the distribution of the (m, z^*)-approximant r.v. and the r.v. X, we would need at least $2m$ parameters, thus our approximation is optimal in this sense.

Remark 3. In general one can not take $z^* = 0$ in Theorem 1. This is due to the fact that not all distributions are determined by their moments (for example, it is well-known that the log-normal distribution is not uniquely determined by its moments). Therefore the use of the Esscher transform is unavoidable in our method.

Proof of Theorem 1. Without loss of generality we may assume that $a = 0$ in (10). Let $U(\mathrm{d}t)$ be the Thorin measure as appears in (10). We define the two functions

$$\psi(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\ln(\phi_X(z)) = \int_0^\infty \frac{U(\mathrm{d}t)}{t+z}, \tag{14}$$

and $\psi^*(z) = \psi(z + z^*)$ for $z > 0$. The function ψ is analytic in $\mathbb{C} \setminus (-\infty, 0]$, thus ψ^* is analytic in $\mathbb{C} \setminus (-\infty, -z^*]$. After the change of variables $w = 1/(t + z^*)$ in (14), the function ψ^* can be written in the form

$$\psi^*(z) = \int_0^{1/z^*} \frac{\mu(\mathrm{d}w)}{1 + wz} \tag{15}$$

for a positive measure $\mu(\mathrm{d}w)$ on $(0, 1/z^*)$, which is simply the pushforward of $U(\mathrm{d}t)$ using the function $w(t) = 1/(t + z^*)$. Integral representation (15) tells us that $\psi^*(z)$ is a Stieltjes function, and it is well-known (Baker and Graves-Morris, 1996) that Stieltjes functions can be approximated very well by certain rational functions, called *Padé approximants*. This is the central idea behind the proof of Theorem 1.

First of all, we note that the function ψ^* is analytic in the disk $|z| < z^*$, thus it can be expanded in Taylor series as follows

$$\psi^*(z) = \sum_{k \geq 0} s_k z^k, \tag{16}$$

where

$$s_k = \frac{\psi^{*(k)}(0)}{k!} = \frac{\psi^{(k)}(z^*)}{k!}. \tag{17}$$

The $[m - 1/m]$ Padé approximation to ψ^* is a rational function of the form $P(z)/Q(z)$, where $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1}$ and $Q(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_m z^m$ are two polynomials that satisfy

$$\frac{P(z)}{Q(z)} - \psi^*(z) = O(z^{2m}) \text{ for } z \to 0. \tag{18}$$

In other words, the first $2m$ coefficients of the Taylor expansion of $P(z)/Q(z)$ at zero should match the corresponding $2m$ coefficients of ψ^*. According to Corollary 1 on page 164 in Baker and Graves-Morris (1996), due to the fact that ψ^* is a Stieltjes function, the $[m - 1/m]$ Padé approximation to ψ^* exists and is unique. Furthermore, Theorem 5.4.1 in Baker and Graves-Morris (1996) (also Theorems 2.2 and 3.1 in Allen et al., 1975) tells us that the denominator $Q(z)$ of the $[m - 1/m]$ Padé approximation has m simple zeros z_i that lie in $(-\infty, -z^*)$ and that the rational function $\psi^*_m(z) := P(z)/Q(z)$ can be written in the partial fraction form as

$$\psi^*_m(z) = \sum_{i=1}^{m} \frac{\alpha_i}{z - z_i}, \tag{19}$$

where $\alpha_i = P(z_i)/Q'(z_i) > 0$, for $i = 1, \ldots, m$. Let us define $\beta_i = -z_i - z^*$ (note that $\beta_i > 0$). We claim that the numbers α_i and β_i thus defined give us the random variable $\widetilde{X}$ (defined via (11)) that satisfies (13). To see this, we check first that

$$\phi_{\tilde{X}_m}(z) = \prod_{i=1}^{m} (1 + z/\beta_i)^{-\alpha_i} = \exp\left(- \int_0^z \psi^*_m(w - z^*) \mathrm{d}w \right). \tag{20}$$

By construction, $\psi^*_m(0) = \psi^*(0)$ and the function $w \mapsto \psi^*_m(w - z^*)$ has the first $2m - 1$ derivatives at z^* equal to the corresponding derivatives of $\psi^*(z - z^*)$. Since

$$\psi^*(z) = -\frac{\mathrm{d}}{\mathrm{d}z} \ln(\phi_X(z + z^*)), \quad \psi^*_m(z) = -\frac{\mathrm{d}}{\mathrm{d}z} \ln(\phi_{\tilde{X}}(z + z^*))$$

we conclude that the first $2m$ derivatives of $\phi_{\tilde{X}}(z)$ at z^* (as defined in (20)) are equal to the corresponding $2m$ derivatives of $\phi_X(z)$ at z^*, and this is equivalent to the moment condition (13).

Let us now prove uniqueness of coefficients α_i and β_i. Assume that we have found numbers $\alpha_i > 0$ and $\beta_i > 0$ such that (13) holds. But then, the first $2m$ derivatives of $\phi_{\tilde{X}}(z)$ at $z = z^*$ are equal to the corresponding $2m$ derivatives of $\phi_X(z)$ at z^*. Reversing the above argument, we conclude that the rational function $-\frac{\mathrm{d}}{\mathrm{d}z}\phi_{\tilde{X}}(z+z^*)$ is the $[m-1/m]$ Padé approximation to $\psi^*(z)$. Thus the uniqueness of this Padé approximation implies the uniqueness (up to permutation) of the coefficients α_i and β_i. □

Proof of Theorem 2. We recall from the above proof of Theorem 1 that the function $\psi^*_m(z)$ is the $[m-1/m]$ Padé approximation to Stieltjes function $\psi^*(z)$. Theorem 5.4.4 in Baker and Graves-Morris (1996) states that $\psi^*_m(z) \to \psi^*(z)$ as $m \to +\infty$, exponentially fast on compact subsets of $\mathbb{C} \setminus (-\infty, -z^*]$. This fact implies that as $m \to +\infty$ the functions $\phi_{\tilde{X}}(z)$ (defined by (20)) converge to

$$\phi(z) = \exp\left(-\int_0^z \psi^*(w - z^*)\mathrm{d}w\right)$$

exponentially fast on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. □

3 The numerical basis for the proposed method

In this section we show how to find the (m, z^*)-approximant to any GGC random variable X with a known p.d.f. $f_X(x)$. Our goal is to compute the unique coefficients $\alpha_i = \alpha_i(m)$ and $\beta_i = \beta_i(m)$, $i = 1, 2, \ldots, m$, whose existence is guaranteed by Theorem 1. We recall that these coefficients give us the (m, z^*)-approximant in the form $\widetilde{X} = \sum_{i=1}^m \Gamma_i$, where $\Gamma_i \sim Ga(\alpha_i, \beta_i)$ are independent r.v.'s.

In the next subsection we present the algorithm, and later we explain why it works and how one could implement it.

3.1 The algorithm for computing α_i and β_i

We organize the algorithm in four steps, so that each step is simple and self-contained and uses only the results of computations from the previous step. These four steps are given below:

Step 1: For $k = 0, 1, \ldots, 2m$, we compute numerically the following integrals

$$g_k := (-1)^k \int_0^\infty x^k e^{-z^* x} f_X(x)\mathrm{d}x. \tag{21}$$

Step 2: We set $s_0 = -g_1/g_0$ and then, for $k = 1, 2, \ldots, 2m-1$, we compute recursively

$$s_k = -\frac{1}{g_0}\left(\frac{g_{k+1}}{k!} + \sum_{i=0}^{k-1} s_i \frac{g_{k-i}}{(k-i)!}\right). \tag{22}$$

Step 3: We solve the $m \times m$ system of linear equations

$$\begin{bmatrix} s_0 & s_1 & s_2 & \cdots & s_{m-1} \\ s_1 & s_2 & s_3 & \cdots & s_m \\ s_2 & s_3 & s_4 & \cdots & s_{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_m & s_{m+1} & \cdots & s_{2m-2} \end{bmatrix} \begin{bmatrix} b_m \\ b_{m-1} \\ b_{m-2} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} s_m \\ s_{m+1} \\ s_{m+2} \\ \vdots \\ s_{2m-1} \end{bmatrix} \tag{23}$$

to find b_i, $1 \le i \le m$. Then we set $a_0 = s_0$ and compute

$$a_k = s_k + \sum_{i=1}^{k} b_i s_{k-i} \tag{24}$$

for $k = 1, 2, \ldots, m-1$.

Step 4: We define polynomials $Q(z) := 1 + b_1 z + b_2 z^2 + \cdots + b_m z^m$ and $P(z) := a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1}$ and compute $\{z_i\}_{1 \le i \le m}$, which are the zeroes of the polynomial $Q(z)$ (it is known that these zeroes are all real, simple and lie in $(-\infty, -z^*)$). The desired coefficients α_i and β_i are given by

$$\beta_i = -z_i - z^* \quad \text{and} \quad \alpha_i = \frac{P(z_i)}{Q'(z_i)} \tag{25}$$

for $i = 1, 2, \ldots, m$.

3.2 Why does this algorithm work?

The algorithm presented in Section 3.1 aims to compute the $[m-1/m]$ Padé approximation (centered at $z = 0$) to the Stieltjes function $\psi^*(z)$, that we introduced in the Proof of Theorem 1. We recall that $\psi^*(z) = -\mathrm{d}/\mathrm{d}z[\ln(\phi_X(z + z^*))]$, where $\phi_X(z) = \mathbb{E}[\exp(-zX)]$. To find the Padé approximation, we need to know the Taylor coefficients of $\psi^*(z)$, which we defined as s_k (see (16)). The goal of Steps 1 and 2 of our algorithm is to compute these coefficients. We start with computing the coefficients g_k, defined by (21). Note that $g_k = \phi_X^{(k)}(z^*)$. Taking the identity $\phi_X'(z + z^*) = -\psi^*(z)\phi_X(z + z^*)$, rewriting it in an equivalent form

$$\sum_{k \ge 0} g_{k+1} z^k / k! = -\left(\sum_{m \ge 0} s_m z^m\right) \times \left(\sum_{k \ge 0} g_k z^k / k!\right).$$

and comparing the coefficients in front of z^k we obtain recursive identity (22) for computing s_k.

Once we have s_k, we can compute the Padé approximation, and this is done by solving the system of linear equations (23). This is a well-known method for finding Padé approximations (e.g. Baker and Graves-Morris, 1996). An interested reader may also derive the system of equations (23) by rewriting condition (18) in the form

$$(a_0+a_1z+a_2z^2+\cdots+a_{m-1}z^{m-1})-\left(\sum_{m\geq 0}s_mz^m\right)(1+b_1z+b_2z^2+\cdots+b_mz^m)=O(z^{2m}) \text{ for } z\to 0. \quad (26)$$

and comparing the coefficients of z^k for $m \leq k \leq 2m-1$. Formulas (24) then follow by comparing the coefficients of z^k for $0 \leq k \leq m-1$ in (26).

The last step of the algorithm computes the coefficients α_i and β_i from the Padé approximant $P(z)/Q(z)$, see the end of the proof of Theorem 1.

3.3 How to implement this algorithm?

While the algorithm is quite straightforward, there is one difficulty in that it requires high-precision arithmetic. High precision is needed in Step 2, as the numbers g_k grow very fast in absolute value and have alternating sign, thus (22) leads to subtracting large numbers and the inevitable loss of precision. High-precision arithmetic is also needed in Step 3, when solving the system of linear equations, as the matrix may be ill-conditioned, and, of course, it is absolutely necessary in Step 4, when finding numerically the roots of the polynomial $Q(z)$.

To facilitate the applications of our algorithm, we have precomputed the coefficients α_i and β_i for all values of $\sigma \in \{0.05, 0.06, \ldots, 2.99, 3\}$, approximation orders $m \in \{10, 20, 30, 40\}$, and $z^* = 1$; these can be downloaded at `www.math.yorku.ca/~akuznets/math.html`. In addition, a copy of the Fortran90 code that we used to compute these coefficients is available upon request.

Now we briefly state the main ideas of implementation of our algorithm. For working with high precision numbers we used a Fortran-90 arbitrary precision package MPFUN90. In our computations we found that working precision of between 300 to 500 digits was enough for most of examples, though in some cases when σ and m were large, we used 700 digits. The main issue in implementing our algorithm from Section 3.1 lies in computing accurately the coefficients g_k in Step 1, as these are the basis for all further computations in Steps 3-4. To compute these numbers accurately and efficiently, we used the double-exponential quadrature of Takahasi and Mori (1974). The idea behind this quadrature is that we perform a change of variables $x = u(y) = \exp(y - \exp(-y))$, $y \in (-\infty,\ \infty)$, and approximate the resulting integral by a Riemann sum

$$I_h := h\sum_{l=-\infty}^{\infty} u(lh)^k e^{-z^* u(lh)} f_X(u(lh))(1+e^{-lh})u(lh).$$

For smooth enough functions $f_X(x)$, the rate of convergence of I_h to g_k is very fast (as $h \to 0^+$). For example, when we take f_X to be a log-normal p.d.f. with $\mu = 0$ and $\sigma = 0.5$, we find that taking $h = 0.008$ and truncating the above infinite series to the range $-1000 < l < 1000$ allows us to compute the first forty coefficients g_k with accuracy better than $1.0e - 300$.

Once the coefficients g_k are computed, Steps 2 and 3 are quite simple. For Step 4, where we needed to find the roots of $Q(z)$, we used Newton's method. This is appropriate, since we know that all roots of $Q(z)$ are real.

3.4 How to choose z^*?

At the outset, recall that the algorithm sketched in Section 3 has two free parameters, which are $m \in \mathbb{N}$ and $z^* > 0$. The former parameter has a simple intuitive interpretation, that is larger values of m yield more accurate approximations. The impact of the latter parameter is harder to describe. We have run several numerical experiments in this respect, computing the maximum absolute difference between the approximating and explicit c.d.f.'s for a large range of values of $z^* > 0$, and our conclusions are that the choice of z^* does not seem to affect the accuracy of the approximation, unless extremely large or extremely small values are employed. For example, when considering the (m, z^*)-approximant $\widetilde{L}$ of $L \sim LN(0.83^2)$ and comparing their c.d.f's, we found that the best accuracy was typically achieved for $z^* \in [0.3,\ 5]$, and the error does not change significantly for the values of z^* in this interval. At the same time, for any $c > 0$, one can show that the $(m, z^*/c)$-approximant $\widetilde{X}$ of the r.v. cX has the same distribution as c times the (m, z^*)-approximant of the r.v. X, thus the good choice of z^* should be rescaled as $z^* \mapsto z^*/c$ when we scale the random variable $X \mapsto cX$. We feel that the choice of $z^* = 1/\mathbb{E}[X]$ is reasonable, however in all numerical examples in this paper we simply took $z^* = 1$.

4 Performance of the approximation algorithm

4.1 Example 1: approximating a single log-normal random variable

We briefly elucidate the accuracy of the approximation algorithm in the case of a single log-normally distributed r.v. $L \sim LN(0.83)$. A few notes are instrumental at the moment. First, the value of the shape parameter $\sigma = 0.83$ has been chosen in line with the empirical evidence reported in O'Neill and Wells (1972) in the context of the collision claim payments involving 30 – 40 year old drivers. Second, starting off with the approximation of a single log-normally distributed r.v. allows us to use its c.d.f. as a benchmark of the appropriateness of our approximation (in the case of the sum S_N, we have no explicit expressions to compare to). Last but not least, as we have already emphasized, the choice of the location and scale parameters being equal to 0 and 1, respectively is made for convenience only, and the inclusion

	$x = 0.85$	$x = 0.9$	$x = 0.91$	$x = 0.92$
$m = 2$	3.01701656990E-008	1.63177011549E-004	5.95571826199E-004	1.91150135288E-003
$m = 4$	3.00610295544E-008	1.63142901323E-004	5.95527541587E-004	1.91148724440E-003
$m = 6$	3.00610413228E-008	1.63142901488E-004	5.95527541680E-004	1.91148724406E-003
$m = 8$	3.00610124570E-008	1.63142901459E-004	5.95527541661E-004	1.91148724404E-003
$\tilde{F}(16x)$	3.031E-008	1.632E-004	5.956E-004	1.912E-003

Table 1: Approximating the c.d.f. $\mathbb{P}(S_{16} \leq 16x)$, where S_{16} is the sum of sixteen independent and identicallly log-normally distributed r.v.'s with $\sigma = 0.125$. We denote by $\tilde{F}(16x)$ the results from Asmussen et al. (2016b)[Table 1].

of the general three parameter log-normal distribution (e.g., O'Neill and Wells, 1972) is straightforward.

We set $z^* = 1$ and evoke the approximation algorithm with $m \in \{10,\ 20,\ 30,\ 40\}$ and then compute the c.d.f. of the respective approximant r.v.'s via the inversion of the corresponding approximating Laplace transforms (see, Appendix A) (e.g., Hürlimann, 2001, for an alternative approach). We choose large number of discretization points when computing the inverse Laplace transforms and ensure that the errors from this step are less than $1.0e-12$ (to ensure that the errors are less than $1.0e-12$ we increase the number of discretization points even further and check that the change in the results is less than $1.0e-12$). Then we compare the approximating c.d.f.'s with the explicit c.d.f. The outcomes are depicted in Figure 1. Remarkably, the figure suggests that the approximation error in the right tail is only visible for $m = 10$, and the approximating c.d.f.'s are not visually distinguishable from the original c.d.f. even for this small value of m.

4.2 Example 2: approximating a sum of 16 log-normal random variables

Next, we consider a numerical example from Asmussen et al. (2016b). The goal is to compute the c.d.f of the sum $S_{16} = L_1 + L_2 + \cdots + L_{16}$, where L_i are i.i.d. random variables distributed log-normally with $\sigma = 0.125$. The results are presented in Table 1. We see that our approximations converge very rapidly, and with $m = 8$ we obtain accuracy of at least 10^{-12}. For comparison, in the last row we present the results from Asmussen et al. (2016b). These results were compared in Asmussen et al. (2016b) with Monte Carlo approximations, and were found to have relative error of about 1%.

We repeat the above example, but for the right tail and for a larger value of $\sigma = 1.5$. The results are presented in Table 2. Our approximations still converge, but at a somewhat slower rate. That said, the obtained approximated c.d.f.'s are still quite accurate, as is seen from the comparison with the Monte

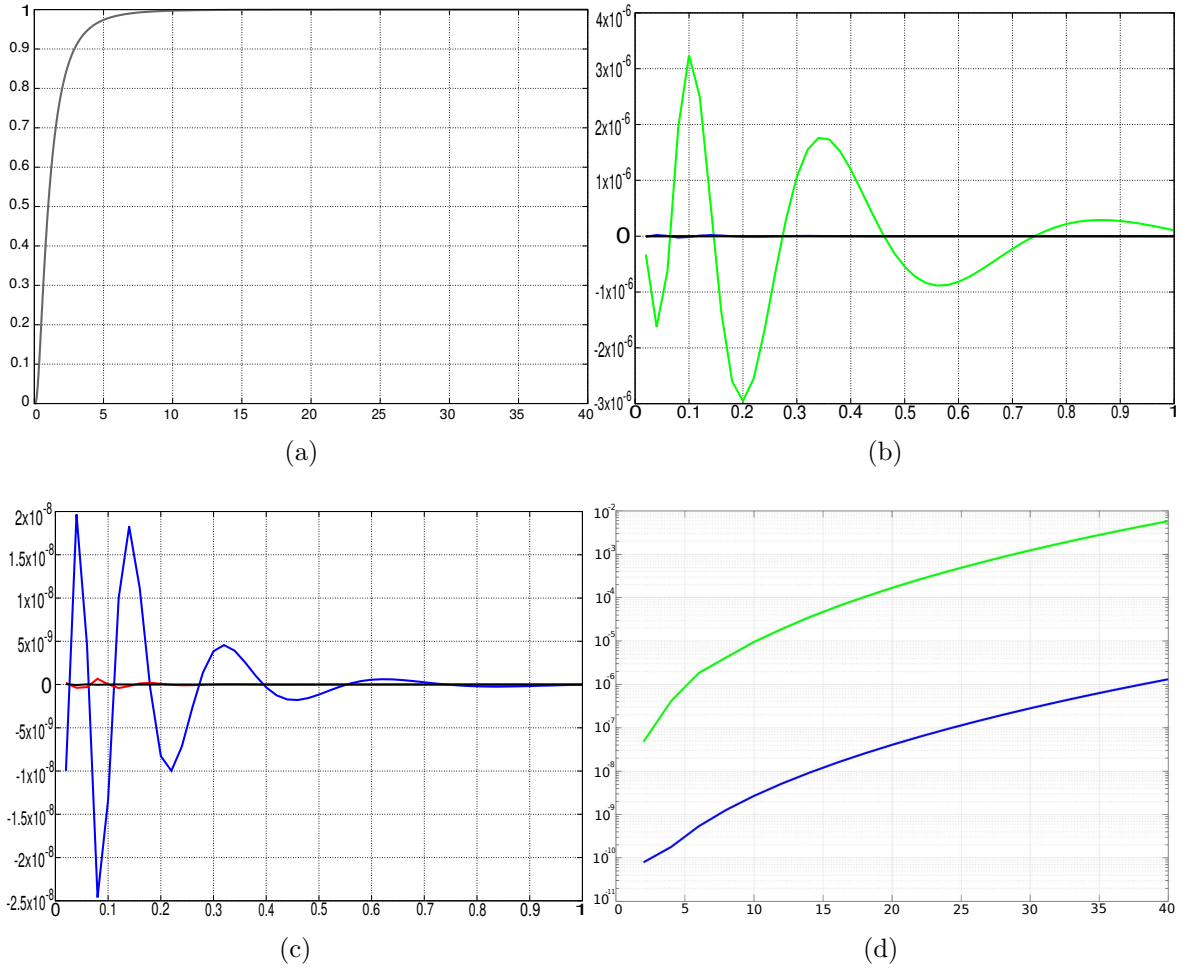


Figure 1: Whenever it applies, the colours: green, blue, red, and black correspond to the approximations of order $m = 10$, 20, 30, and 40, respectively. (a) The c.d.f. F of $L \sim LN(0.83)$ compared with the c.d.f.'s $F_{\widetilde{L}}$ of the approximant r.v.'s $\widetilde{L}$. (b) The errors $F(x) - F_{\widetilde{L}}(x)$ in the left tail. (c) Same as in panel (b) but omitting the $m = 10$ (green) approximation. (d) The errors $F(x) - F_{\widetilde{L}}(x)$ in the right tail.

	$x = 12$	$x = 25$	$x = 40$	$x = 60$
$m = 10$	0.99283646	0.99967148	0.99998468	0.99999968
$m = 20$	0.99216586	0.99925807	0.99985210	0.99997206
$m = 30$	0.99214659	0.99923867	0.99983322	0.99995759
$m = 40$	0.99214492	0.99923699	0.99983154	0.99995591
$\hat{F}(x)$	0.99214	0.99924	0.99983	0.99995

Table 2: Approximating the c.d.f. $\mathbb{P}(S_{16} \leq 16x)$, where S_{16} is the sum of sixteen independent and identically log-normally distributed r.v.'s with $\sigma = 1.5$. We denote by $\hat{F}(16x)$ the Monte Carlo estimate based on a sample of size 10^9.

Carlo - based approximations (last raw of Table 2).

5 Aggregate economic capital determination and allocation: Individual Risk Model

We have already mentioned that the c.d.f. of a sum, random or not, of independent and log-normally distributed r.v.'s can be approximated with the help of Theorem 1 via Equation (8). Remarkably, risk measures (3) and (4), as well as the allocation rules based on them, can be approximated in a very similar fashion. Speaking briefly, the reason for this lies in the connection between the aforementioned risk measures and allocations and the notion of the size-biased distributions (Patil and Rao, 1978), which are briefly introduced next.

Let X be a positive r.v., then for $l \in \mathbb{N}$, such that $\mathbb{E}[X^l] < +\infty$, the size-biased r.v. of order l is denoted by $X^{*(l)}$ and defined via the distribution function

$$\mathbb{P}\left(X^{*(l)} \in \mathrm{d}x\right) = \frac{x^l}{\mathbb{E}[X^l]}\mathbb{P}\left(X \in \mathrm{d}x\right) \text{ for } x \geq 0 \tag{27}$$

(e.g., Patil and Rao, 1978). For $l = 1$, we simplify the notation and write X^*, F_{X^*}, and ϕ_{X^*} for the size-biased variant of X, its c.d.f., and Laplace transform.

When both the original and the size-biased c.d.f.'s belong to the same family of c.d.f.'s, we say that the distribution is closed under size-biasing of order $l \in \mathbb{N}$. It is not difficult to see that the log-normal distribution is closed under size-biasing of order one (Patil and Rao, 1978), and consequently the expression for the CTE risk measure for the r.v. L is, for $q \in [0,\ 1)$,

$$\mathrm{CTE_q[L]} = \mathbf{E}\left[\mathrm{L}|\ \mathrm{L} > \mathrm{VaR_q[L]}\right] = \frac{\mathbf{E}[\mathrm{L}]}{1-\mathrm{q}}\mathbb{P}\left[\mathrm{L}^* > \mathrm{VaR_q[L]}\right],$$

(Furman and Zitikis, 2008b), where $L^* \sim LN(\sigma^2, \sigma^2)$ and, if needed, Equation (8) can be evoked.

More generally, it can be shown that the log-normal distribution is closed under size-biasing of any order, and so risk measure (4), as well as other ones based on higher order moments, are trivial to compute for the r.v. L. That said, computing expressions for risk measures (3) and (4) for the aggregate r.v. S_n, $n \in \mathbb{N}$ is substantially more involved, and has to deal with the size-biasing of the r.v. $S_n = \sum_{i=1}^n L_i$, $L_i \sim LN(\sigma_i^2)$. This is achieved in the next theorem, which shows that the Laplace transform of the l-th order size-biased variant of the r.v. S_n can be spelled out in terms of the Laplace transforms of the summands $L_1, \ldots, L_n$. As a result approximations akin to (8) are feasible.

We note at the outset that the Laplace transform of the size-biased r.v. $X^{*(l)}$, such that $\mathbb{E}[X^l] < +\infty$, is given by

$$\phi_{X^{*(l)}}(z) := \mathbb{E}[\exp(-zX^{*(l)})] = \frac{\mathbb{E}[X^l e^{-zX}]}{\mathbb{E}[X^l]} \text{ for } \mathrm{Re}(z) \geq 0,\ l \in \mathbb{N}. \tag{28}$$

The next theorem allows to reduce the problem of computing the c.d.f. of the r.v. S_n to a remarkably more tractable set-up of finite sums of independent gamma distributed r.v.'s.

Theorem 3. *Let $L \sim LN(\sigma^2)$, $\sigma > 0$ be a log-normally distributed r.v. with the corresponding Laplace transform ϕ_L. Also, let $L_j \sim LN(\sigma_j)$ be mutually independent and log-normally distributed r.v.'s with shape parameters $\sigma_j > 0$ and the corresponding Laplace transforms ϕ_{L_j}, $j = 1, \ldots, n$. Finally, let $S_n = \sum_{j=1}^n L_j$ as before. Then the following assertions hold:*

(1) For any $l \in \mathbb{N}$ and $c = \exp(l\sigma^2)$, we have

$$\phi_{L^{*(l)}}(z) = \phi_{cL}(z) \text{ for } \mathrm{Re}(z) \geq 0. \tag{29}$$

(2) For any $l \in \mathbb{N}$ and $c_j = \exp(d_j \sigma_j^2)$, $j = 1, \ldots, n$, we have that the Laplace transform of the size-biased variant of order l of the sum S_n is the following weighted average of Laplace transforms

$$\phi_{S_n^{*(l)}}(z) = \frac{1}{\mathbb{E}[S^l]} \sum_{d_1 + \cdots + d_n = l} \binom{l}{d_1, \ldots, d_n} \prod_{j=1}^n \mathbb{E}\left[L_j^{d_j}\right] \phi_{c_1 L_1 + \cdots + c_n L_n}(z) \text{ for } \mathrm{Re}(z) \geq 0. \tag{30}$$

Proof. It is not difficult to check that, for $l \in \mathbb{N}$, we have $L_j^{*(l)} \sim LN(l\sigma^2, \sigma^2)$, which proves (29). To

confirm (30), we have the following string of equations

$$\begin{aligned}
\phi_{S_n^{*(l)}}(z) &= \frac{\mathbb{E}[S_n^l e^{-zS_n}]}{\mathbb{E}[S_n^l]} = \frac{1}{\mathbb{E}[S_n^l]}\mathbb{E}\left[\left(\sum_{j=1}^n L_j\right)^l e^{-zS_n}\right] \\
&= \frac{1}{\mathbb{E}[S_n^l]}\sum_{d_1+\cdots+d_n=l}\binom{l}{d_1,\ldots,d_n}\mathbb{E}\left[\prod_{j=1}^n L_j^{d_j} e^{-zL_j}\right] \\
&= \frac{1}{\mathbb{E}[S_n^l]}\sum_{d_1+\cdots+d_n=l}\binom{l}{d_1,\ldots,d_n}\prod_{j=1}^n \mathbb{E}\left[L_j^{d_j}\right]\prod_{j=1}^n \frac{\mathbb{E}\left[L_j^{d_j} e^{-zL_j}\right]}{\mathbb{E}[L_j^{d_j}]} \\
&= \frac{1}{\mathbb{E}[S_n^l]}\sum_{d_1+\cdots+d_n=l}\binom{l}{d_1,\ldots,d_n}\prod_{j=1}^n \mathbb{E}\left[L_j^{d_j}\right]\phi_{L_1^{*(d_1)}+\cdots+L_n^{*(d_n)}}(z) \\
&= \frac{1}{\mathbb{E}[S_n^l]}\sum_{d_1+\cdots+d_n=l}\binom{l}{d_1,\ldots,d_n}\prod_{j=1}^n \mathbb{E}\left[L_j^{d_j}\right]\phi_{c_1L_1+\cdots+c_nL_n}(z),
\end{aligned}$$

which completes the proof. □

Theorem 3 implies that, for any $l \in \mathbb{N}$, we can approximate the Laplace transforms of the p.d.f.'s of the r.v.'s $L_j^{*(l)}$ and $S_n^{*(l)}$ using Theorem 1, and this is precisely what we need latter on in this section. We note in passing that a particularly simple special case of (30) occurs for $l = 1$. Then we readily have, for $c_j = \exp(\sigma_j^2)$,

$$\phi_{S_n^*}(z) = \sum_{j=1}^n \frac{\mathbb{E}[L_j]}{\mathbb{E}[S_n]}\phi_{c_jL_j+\sum_{i=1,i\neq j}^n L_i}(z) \text{ for } \mathrm{Re}(z) \geq 0 \tag{31}$$

(e.g., Furman and Landsman, 2005, Proposition 1).

In the rest of this section we aim at answering the following question: how much Economic Capital (EC) is required to support the risk S_n? To this end, let $H : \mathcal{X} \to [0, \infty]$ denote a regulatory risk measure that maps risk r.v.'s in the set of actuarial risks $\mathcal{X}$ to EC's in the extended non-negative half-line. Nowadays the determination of the aggregate EC - $H[S_N]$ - is a compulsory task for insurers (e.g., Solvency II, Swiss Solvency Test). One of the most popular risk measures employed for this purpose is the conditional tail expectation

$$\mathrm{CTE}_q[S_n] = \mathbb{E}[S_n |\ S_n > \mathrm{VaR}_q[S_n]] \text{ for } S_n \text{ with finite mean}, \tag{32}$$

where $\mathrm{VaR}_q[S_n]$ is the Value-at-Risk, and $q \in [0,\ 1)$ is the prudence parameter set by the regulations. We note in passing that, for risk r.v.'s with continuous c.d.f.'s, the CTE risk measure coincides with the Expected Shortfall (ES) risk measure, and so it is coherent in the sense of Artzner et al. (1999) (also, Hürlimann, 2003; McNeil et al., 2005, Lemma 2.16), and belongs to the class of distorted (Wang, 1996) and weighted (Furman and Zitikis, 2008b) risk measures.

If the variability along the right tail is of a concern, the modified Tail Variance (mTV) risk measure of Furman and Landsman (2006) may become useful (also, Jiang et al., 2016, for a recent application). The mTV takes into account both the magnitude and the variability of the tail risk, and for the aggregate risk S_n it is defined as

$$\mathrm{mTV}_q[S_n] = \mathbb{E}[S_n|\ S_n > \mathrm{VaR}_q[S_n]] + \frac{\mathbf{Var}[S_n|\ S_n > \mathrm{VaR}_q[S_n]}{\mathbb{E}[S_n|\ S_n > \mathrm{VaR}_q[S_n]]} \text{ for } S_n \text{ with finite variance,} \tag{33}$$

where $q \in [0,\ 1)$ is the prudence parameter.

Our ultimate goal is to approximate (32) and (33). To this end, we choose $n = 3$ and assume that the summands of S_n represent log-normally distributed risks due to three business lines of an insurer. We then derive the desired approximations using the algorithm described in Section 3 with $m = 20$. Since the order of the approximation does not vary any more, we use the "tilde" notation with no subscripts for all the approximating quantities below.

Let $p_i = \exp(\sigma_i^2/2),\ i = 1, \ldots, n$, and $p_+ = \sum_{i=1}^n p_i$. Also, note that, for S_n denoting a sum of independent r.v.'s, we have

$$\mathbb{E}[S_n^l \mathbf{1}\{S_n > s\}] = \mathbf{E}[S_n^l] \left(1 - F_{S_n^{*(l)}}(s)\right),\ s \geq 0, \tag{34}$$

where all the involved quantities are assumed well-defined and finite.

Example 1. We start with the general IRM, that is the r.v. S_n, with log-normally distributed summands. Let $e(s) := \mathbb{E}[S_n \mathbf{1}\{S_n > s\}],\ s \geq 0$. Then, using Equations (34) and (8), we have

$$e(s) = \mathbb{E}[S_n] \mathcal{L}^{-1} \left\{ \frac{1 - \phi_{S_n^*}(z)}{z} \right\} (s) \text{ for } s \geq 0. \tag{35}$$

Consequently, for $q \in [0,\ 1)$, we obtain with the help of Theorem 3,

$$\begin{aligned} \mathrm{CTE}_q[S_n] &= \frac{p_+}{1-q} \mathcal{L}^{-1} \left\{ \frac{1 - \phi_{S_n^*}(z)}{z} \right\} (\mathrm{VaR}_q[S_n]) \\ &= \frac{p_+}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left(1 - \sum_{i=1}^n \frac{p_i}{p_+} \phi_{c_i L_i + \sum_{j=1, j\neq i}^n L_j}(z) \right) \right\} (\mathrm{VaR}_q[S_n]). \end{aligned}$$

Therefore, for $n = 3$, $c_i = \exp(\sigma_i^2)$, and $q \in [0,\ 1)$, we have the following approximation

$$\mathrm{CTE}_q[S_3] \approx \frac{p_+}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left(1 - \sum_{i=1}^3 \frac{p_i}{p_+} \phi_{c_i \widetilde{L}_i + \sum_{j=1, j\neq i}^3 \widetilde{L}_j}(z) \right) \right\} (\mathrm{VaR}_q[\widetilde{S}_3]), \tag{36}$$

where $\widetilde{L}_i$ and $\widetilde{S}_3$ are obtained with the help of Theorem 1. This establishes the desired approximation.

To illustrate, assume that the r.v.'s $L_1,\ L_2$ and L_3 are distributed $LN(0.83)$. We depict (36) as well as the CTE of the comonotonic sum $S_3^c := 3L_{0.83}$ in Figure 2a. We note in passing that the two are

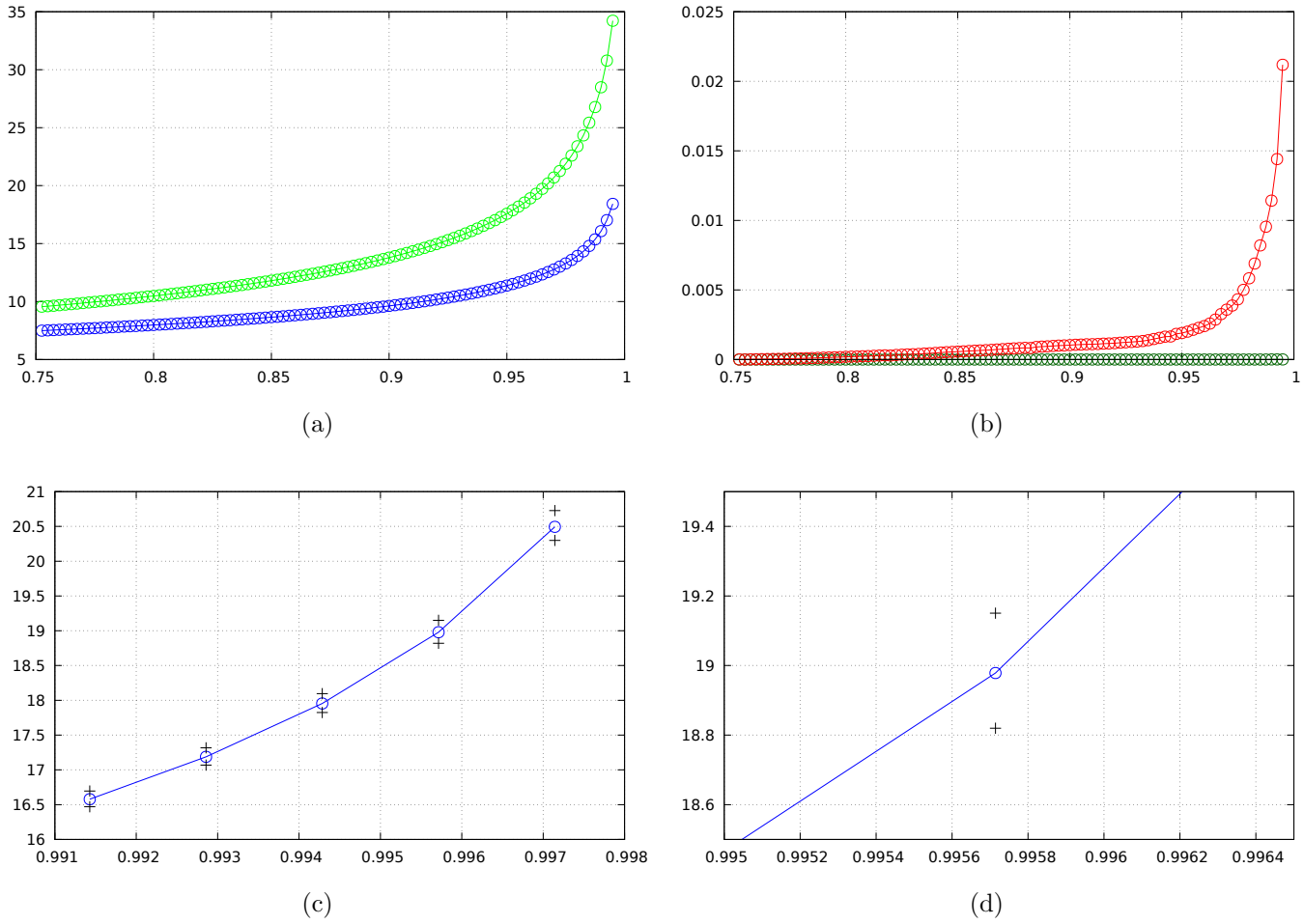


Figure 2: (a) Degree $m = 20$ approximations $\mathrm{CTE}_q[\widetilde{S_3^c}]$ (green) and $\mathrm{CTE}_q[\widetilde{S}_3]$ (blue), $\sigma_1 = \sigma_2 = \sigma_3 = 0.83$. (b) The absolute errors between $\mathrm{CTE}_q[S_3]$ as computed by numerical integration and: (i) $\mathrm{CTE}_q[\widetilde{S}_3]$ (dark green); (ii) the average of 50 MC derived values of $\mathrm{CTE}_q[S_3]$ (red) (c) $\mathrm{CTE}_q[\widetilde{S}_3]$ (blue line) compared to the maximum and minimum of $\mathrm{CTE}_q[S_3]$ computed using 50 MC simulations. (d) A close-up of (c) at one point. All simulations are generated using 10^6 random samples.

the lower and the upper bounds, respectively, in the Fréchet set of all joint c.d.f.'s with fixed $LN(0.83)$ margins and varying positive cumulative dependence (Denuit et al., 2001). We compare our approach with the outcomes due to the Monte Carlo (MC) simulation method (50 MC simulations with 10^6 random samples) in Figure 2b (see, also, Figures 2c). This completes Example 1.

In the next example we generalize Example 1 by considering (i) log-normally distributed risks that are not identically distributed, and (ii) a risk measure that quantifies the variability of the tail risk.

Example 2. As before, we start by considering the aggregate r.v. within the IRM framework with log-normally distributed constituents. Let $e_2(s) := \mathbb{E}[S_n^2 \mathbf{1}\{S_n > s\}]$, $s \geq 0$, then by (34) and (8), we readily obtain

$$e_2(s) = \mathbb{E}[S_n^2](1 - F_{S_n^{*(2)}}(s)) = \mathbb{E}[S_n^2]\mathcal{L}^{-1}\left\{\frac{1 - \phi_{S_n^{*(2)}}(z)}{z}\right\}(s) \text{ for } s \geq 0. \tag{37}$$

Hence, for $q \in [0,\ 1)$, we have

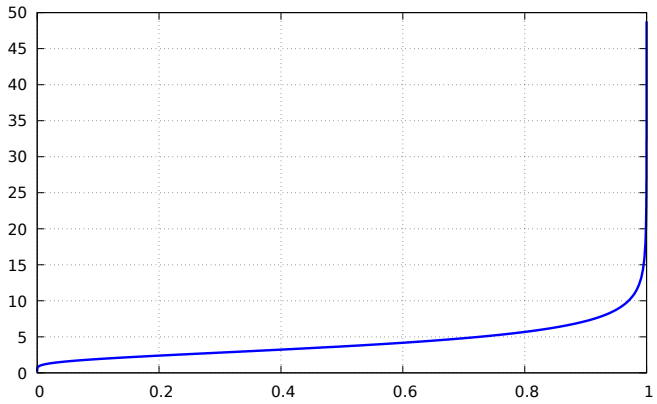
$$\mathbb{E}[S_n^2|\ S_n > \mathrm{VaR}_q[S_n]] \ = \ \frac{\mathbb{E}[S_n^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1 - \phi_{S_n^{*(2)}}(z)}{z}\right\}(\mathrm{VaR}_q[S_n]),$$

which can be simplified further using Theorem 3. Indeed for the special case of interest herein, that is, for $n = 3$, $S_3 = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are independent but not necessarily identically log-normally distributed risks, we have

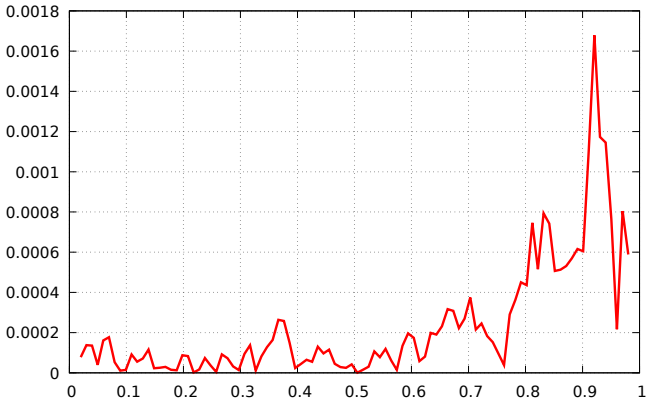
$$\begin{aligned}
\mathbb{E}[S_3^2]\phi_{S_3^{*(2)}}(z) \ &= \ \mathbb{E}[L_1^2]\phi_{c_1L_1+L_2+L_3}(z) + \mathbb{E}[L_2^2]\phi_{L_1+c_2L_2+L_3}(z) + \mathbb{E}[L_3^2]\phi_{L_1+L_2+c_3L_3}(z) \\
&+ \ 2\mathbb{E}[L_1]\mathbb{E}[L_2]\phi_{c_1L_1+c_2L_2+L_3}(z) + 2\mathbb{E}[L_1]\mathbb{E}[L_3]\phi_{c_1L_1+L_2+c_3L_3}(z) \\
&+ \ 2\mathbb{E}[L_2]\mathbb{E}[L_3]\phi_{L_1+c_2L_2+c_3L_3}(z) \text{ for } \mathrm{Re}(z) \geq 0,
\end{aligned}$$

which is formulated in terms of the Laplace transforms of log-normal convolutions, and thus can be approximated with the help of Theorem 1. Bearing this, as well as (36), in mind, we immediately obtain the approximation for the mTV risk measure.

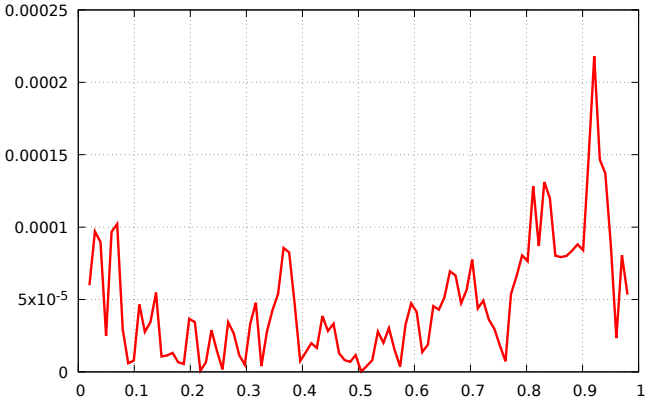
Set $\sigma_1 = 0.81$, $\sigma_2 = 0.83$, and $\sigma_3 = 0.85$ (these choices of parameters are motivated by the empirical findings in O'Neill and Wells (1972)). The approximations $\mathrm{VaR}_q[\widetilde{S}_3]$, $\mathrm{CTE}_q[\widetilde{S}_3]$, and $\mathrm{mTV}_q[\widetilde{S}_3]$ are presented in Figures 3a, 4a, and 4e, respectively. In Figures 3b and 4b we compute the absolute errors between the just-mentioned approximating risk measures and those computed with the help of the MC approach (average of 50 MC simulated values of VaR, CTE, and mTV, each generated with 10^6 random samples). This completes Example 2.



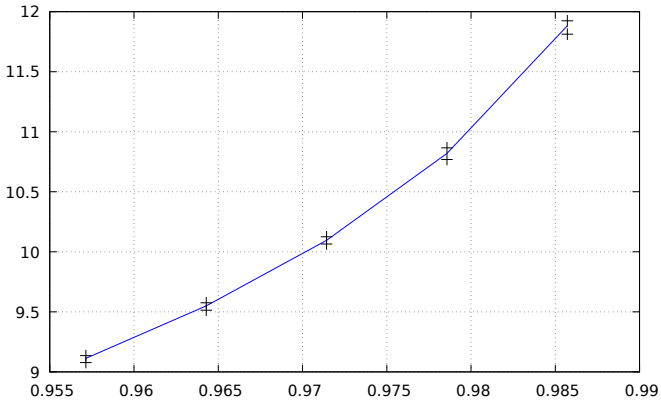
(a)



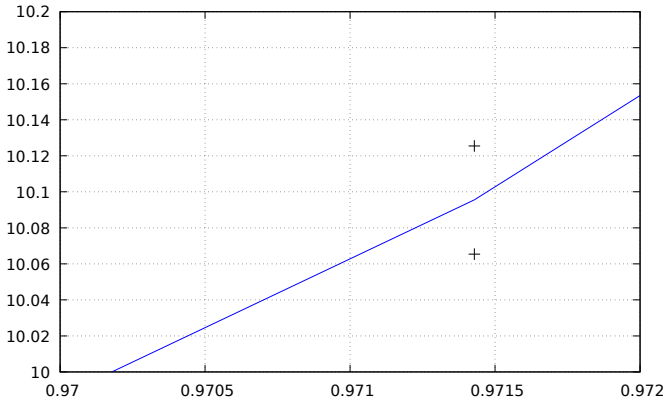
(b)



(c)

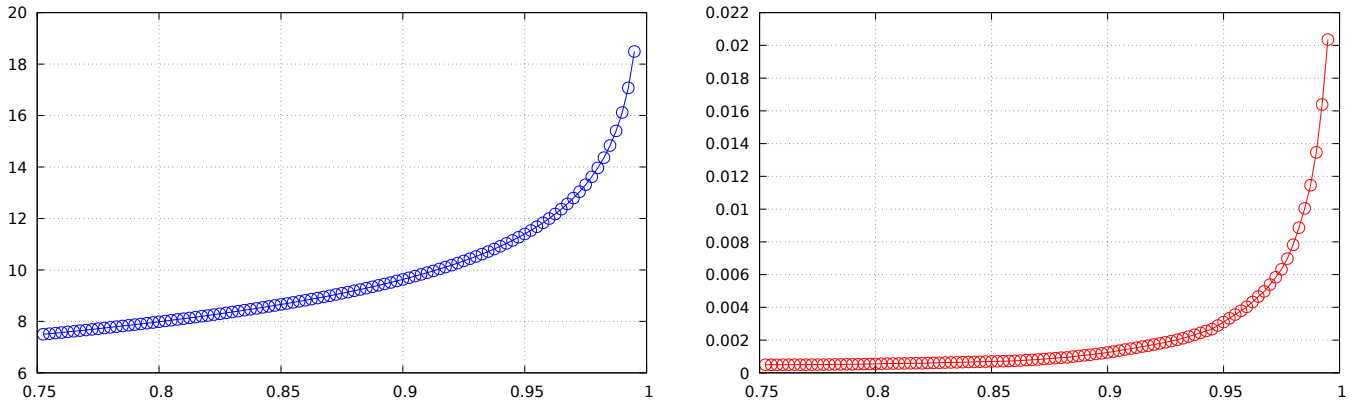


(d)



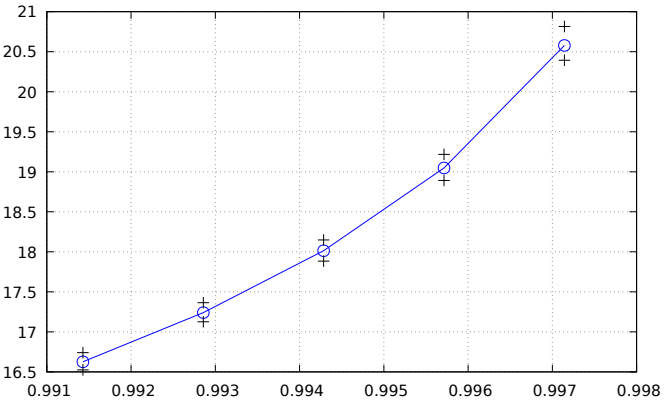
(e)

Figure 3: (a) Degree $m = 20$ approximation $\mathrm{VaR}_q[\widetilde{S}_3]$, where $\sigma_1 = 0.81$, $\sigma_2 = 0.83$ and $\sigma_3 = 0.85$. (b) Absolute error between $\mathrm{VaR}_q[\widetilde{S}_3]$ and the average of 50 MC simulations of $\mathrm{VaR}_q[S_3]$. (c) Relative absolute error between $\mathrm{VaR}_q[\widetilde{S}_3]$ and the average of 50 MC simulations of $\mathrm{VaR}_q[S_3]$. (d) $\mathrm{VaR}_q[\widetilde{S}_3]$ (blue line) compared to the maximum and minimum of $\mathrm{VaR}_q[S_3]$ computed using 50 MC simulations. (e) A close-up of (d) at one point. All simulations are generated using 10^6 random samples.

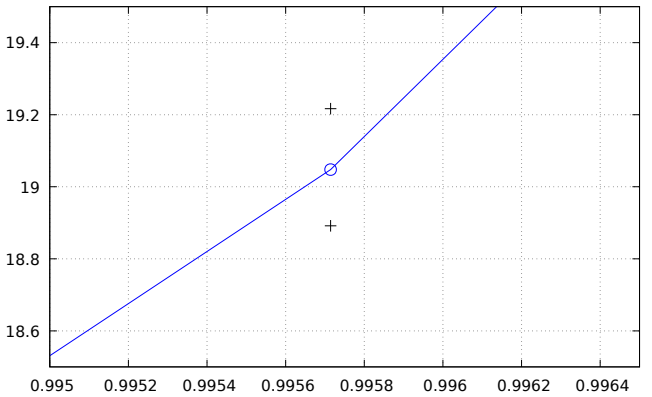


(a)

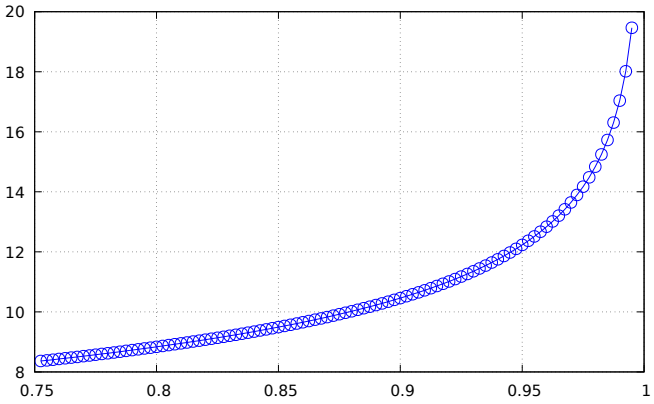
(b)



(c)



(d)



(e)

Figure 4: (a) Degree $m = 20$ approximation $\text{CTE}_q[\widetilde{S}_3]$, where $\sigma_1 = 0.81$, $\sigma_2 = 0.83$ and $\sigma_3 = 0.85$. (b) Absolute error between $\text{CTE}_q[\widetilde{S}_3]$ and the average of 50 MC simulations of $\text{CTE}_q[S_3]$. (c) $\text{CTE}_q[\widetilde{S}_3]$ (blue line) compared to the maximum and minimum of $\text{CTE}_q[S_3]$ computed using 50 MC simulations. (d) A close-up of (c) at one point. (e) Degree $m = 20$ approximation $\text{mTV}_q[\widetilde{S}_3]$ where $\sigma_1 = 0.81$, $\sigma_2 = 0.83$ and $\sigma_3 = 0.85$. All simulations are generated using 10^6 random samples.

Given that the aggregate EC $H[S_n]$ has been determined, a somewhat more involved problem is the allocation of this EC to constituents. Namely, it is often of interest to assess the risk contribution of each summand in $S_n = X_1 + \cdots + X_n$ to $H[S_n]$. More formally, the allocation rule $A : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ such that $A[X, X] = H[X]$ for $X \in \mathcal{X}$ assigns a finite (or infinite) value of the allocated EC to random pairs (X, S) (e.g., Denault, 2001; Furman and Zitikis, 2008a). One goal of the allocation exercise is profitability testing, others are cost sharing and pricing (e.g., Venter, 2004).

Clearly, there are infinitely many ways to allocate the aggregate EC, and the literature on the allocation rules is vast and growing rapidly (e.g., Dhaene et al., 2012, and references therein). In this paper we work with the allocation counterparts of risk measures (32) and (33), which are, respectively,

$$\mathrm{CTE}_q[X_i, S_n] = \mathbb{E}[X_i |\ S_n > \mathrm{VaR}_q[S_n]], \tag{38}$$

and

$$\mathrm{mTCoV}_q[X_i, S_n] := \mathbb{E}[X_i |\ S_n > \mathrm{VaR}_q[S_n]] + \frac{\mathbf{Cov}[X_i,\ S_n |\ S_n > \mathrm{VaR}_q[S_n]}{\mathbb{E}[S_n |\ S_n > \mathrm{VaR}_q[S_n]]}, \tag{39}$$

where $i = 1, \ldots, n$, $q \in [0,\ 1)$, and we assume that all the involved quantitites are well-defined and finite. Obviously, we have $\mathrm{CTE}_q[S_n,\ S_n] = \mathrm{CTE}_q[S_n]$ and $\mathrm{mTCoV}_q[S_n, S_n] = \mathrm{mTV}_q[S_n]$ for $i = 1, \ldots, n,\ q \in [0,\ 1)$, and allocation rules (38) and (39) are fully additive. Moreover, these allocation rules are 'weighted' (Furman and Zitikis, 2008a), and they are optimal in the sense of Dhaene et al. (2012).

We are now ready to delve into the approximation of allocation rules (38) and (39). Beforehand, let $S_{n,-j} := S_n - X_j,\ j = 1, \ldots, n$, then, for non-negative constants l and k, we have

$$\mathbb{E}[X_j^l S_{n,-j}^k \mathbf{1}\{S_n > s\}] = \mathbf{E}[X_j^l]\mathbf{E}[S_{n,j}^l] \left(1 - F_{S_{n,-j}^{*(k)} + X_j^{*(l)}}(s)\right),\ s \geq 0, \tag{40}$$

where we assume that all the involved quantities are well-defined and finite.

Example 3. As in all previous examples, we consider a general IRM with log-normally distributed standalone risks first, and we specialize thereafter. Let $h_0(s) := \mathbb{E}[L_j \mathbf{1}\{S_n > s\}],\ s \geq 0$, then we have, with the help of Equations (40) and (8),

$$h_0(s) = \mathbb{E}[L_j] \mathcal{L}^{-1} \left\{ \frac{1}{z} \left(1 - \phi_{S_n - L_j + L_j^*}(z)\right) \right\}(s) \text{ for } s \geq 0. \tag{41}$$

This implies immediately, for $q \in [0,\ 1)$ and $c_j = \exp(\sigma_j^2)$,

$$\mathrm{CTE}_q[L_j,\ S_n] = \frac{\mathbb{E}[L_j]}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left(1 - \phi_{\sum_{i=1,\ i\neq j}^n L_i + c_j L_j}(z)\right) \right\} (\mathrm{VaR}_q[S_n]).$$

Further, set $n = 3$, then

$$\mathrm{CTE}_q[L_j,\ S_3] \approx \frac{\mathbb{E}[L_j]}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left(1 - \phi_{\sum_{i=1,\ i\neq j}^3 \widetilde{L_i} + c_j \widetilde{L_j}}(z)\right) \right\} (\mathrm{VaR}_q[\widetilde{S_3}]),, \tag{42}$$

which is computed evoking Theorem 1.

In order to approximate the modified Tail Covariance allocation rule, we recall the following trivial equation

$$\begin{aligned}\mathbf{Cov}[X_j,\ S_n|\ S_n > \mathrm{VaR}_q[S_n]] &= \mathbb{E}[X_j S_n|\ S_n > \mathrm{VaR}_q[S_n]] \\ &\quad - \ \mathbb{E}[X_j|\ S_n > \mathrm{VaR}_q[S_n]] \times \mathbb{E}[S_n|\ S_n > \mathrm{VaR}_q[S_n]] \text{ for } q \in [0,\ 1),\end{aligned}$$

where we assume that all the involved quantities are well-defined and finite. The product of expectations can be computed as in Example 1, hence, we only need to approximate the mixed expectation. To this end, let $h(s) := \mathbb{E}[L_j S_n \mathbf{1}\{S_n > s\}],\ s \geq 0$, and note with the help of Equations (40) and (8) that

$$\begin{aligned}h(s) \ &= \ \mathbb{E}[L_j]\mathbb{E}[S_{n,-j}]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S^*_{n,-j}+L^*_j}(z)\right)\right\}(s) \\ &\quad + \mathbb{E}[L_j^2]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S_{n,-j}+L_j^{*(2)}}(z)\right)\right\}(s),\end{aligned}$$

where $S_{n,-j} = \sum_{i=1,\ i\neq j}^{n} L_i$. This can be simplified with the help of Theorem 3, and in particular, for $n = 3$, c_j as before, and $s \geq 0$,

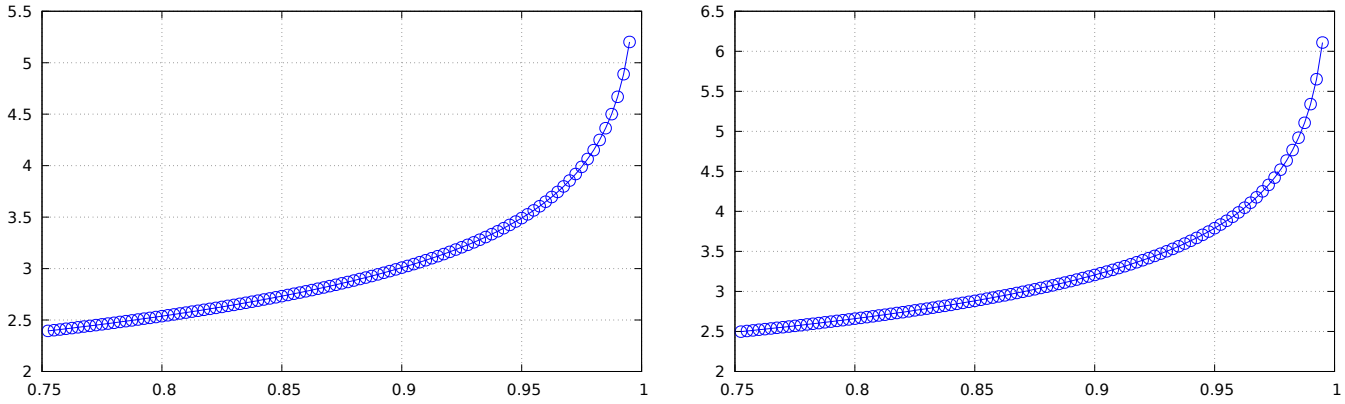
$$\begin{aligned}h(s) \ &= \ \mathbb{E}[L_j]\mathbb{E}[S_{3,-j}]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\sum_{i=1,\ i\neq j}^{3}\frac{\mathbb{E}[L_i]}{\mathbb{E}[S_{3,-j}]}\phi_{c_jL_j+c_iL_i+\sum_{k=1,\ k\neq i,j}^{3}L_k}(z)\right)\right\}(s) \\ &\quad + \mathbb{E}[L_j^2]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S_{3,-j}+c_j^2L_j}(z)\right)\right\}(s),\end{aligned}$$

which implies, for $c_j = \exp(\sigma_j^2)$ and $q \in [0,\ 1)$,

$$\begin{aligned}&\mathbb{E}[L_jS_3|\ S_3 > \mathrm{VaR}_q[S_3]] \qquad (43)\\ &= \ \frac{\mathbb{E}[L_j]\mathbb{E}[S_{3,-j}]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\sum_{i=1,\ i\neq j}^{3}\frac{\mathbb{E}[L_i]}{\mathbb{E}[S_{3,-j}]}\phi_{c_jL_j+c_iL_i+\sum_{k=1,\ k\neq i,j}^{3}L_k}(z)\right)\right\}(\mathrm{VaR}_q[S_3]) \\ &\quad + \frac{\mathbb{E}[L_j^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S_{3,-j}+c_j^2L_j}(z)\right)\right\}(\mathrm{VaR}_q[S_3]) \\ &\approx \ \frac{\mathbb{E}[L_j]\mathbb{E}[S_{3,-j}]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\sum_{i=1,\ i\neq j}^{3}\frac{\mathbb{E}[L_i]}{\mathbb{E}[S_{3,-j}]}\phi_{c_j\widetilde{L}_j+c_i\widetilde{L}_i+\sum_{k=1,\ k\neq i,j}^{3}\widetilde{L}_k}(z)\right)\right\}(\mathrm{VaR}_q[\widetilde{S}_3]) \\ &\quad + \frac{\mathbb{E}[L_j^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{\widetilde{S}_{3,-j}+c_j^2\widetilde{L}_j}(z)\right)\right\}(\mathrm{VaR}_q[\widetilde{S}_3]),\end{aligned}$$

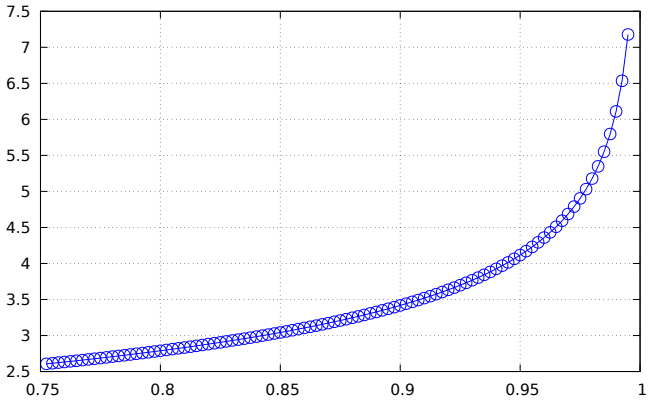
which establishes the approximation of the conditional covariance, and can be computed using Theorem 1. The desired approximation of the modified Tail Covariance follows readily.

For visualization purposes, we again set $\sigma_1 = 0.81,\ \sigma_2 = 0.83$ and $\sigma_3 = 0.85$, then the approximating allocation rules based on the CTE and the modified Tail Covariance risk measures are depicted in Figures 5 and 6. This completes Example 3.



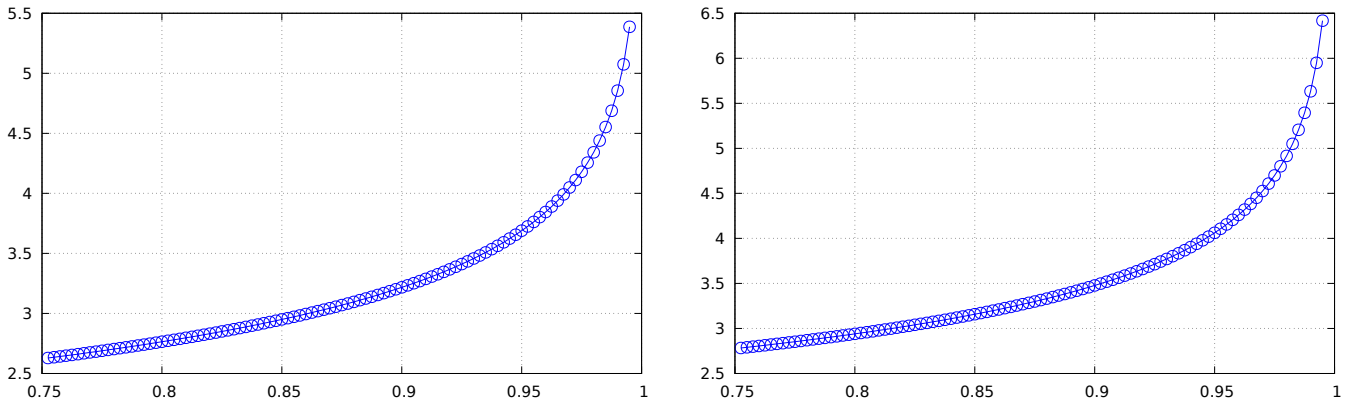
(a)

(b)



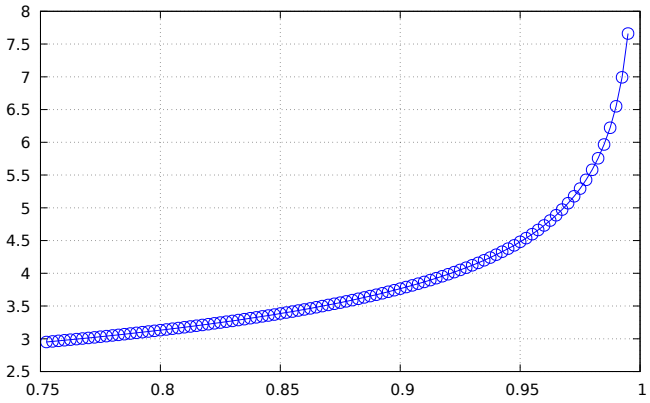
(c)

Figure 5: Degree $m = 20$ approximations of the allocations (a) $\text{CTE}_q[L_1,\ S_3]$ (b) $\text{CTE}_q[L_2,\ S_3]$ and (c) $\text{CTE}_q[L_3,\ S_3]$, where $L_1 \sim LN(0, 81)$, $L_2 \sim LN(0, 83)$, and $L_3 \sim LN(0, 85)$.



(a)

(b)



(c)

Figure 6: Degree $m = 20$ approximations of the allocations (a) $\mathrm{mTCov}_q[L_1,\ S_3]$ (b) $\mathrm{mTCov}_q[L_2,\ S_3]$ (c) $\mathrm{mTCov}_q[L_3,\ S_3]$, where $L_1 \sim LN(0, 81)$, $L_2 \sim LN(0, 83)$, and $L_3 \sim LN(0, 85)$.

6 Aggregate economic capital determination: collective risk model

In this section we assume that N is a discrete r.v., and so $S_N = L_1 + \cdots + L_N$ is a collective risk model with all of $L_1,\ L_2, \ldots$ independent mutually and on N, as well as identically distributed as a canonical r.v. $L \sim LN(\sigma^2),\ \sigma > 0$. Let $G_N(z) = \mathbb{E}[z^N]$ denote the p.g.f. of the r.v. N for all $z \in (-\infty,\ \infty)$ such that the p.g.f. is well-defined and finite.

In principle, once we have fixed the distributions of the r.v.'s L and N, we may proceed exactly as before and approximate the Laplace transform of S_N with the help of Theorem 1. This is clear from

$$\phi_{S_N}(z) = G_N(\phi_L(z)) \approx G_N(\phi_{\widetilde{L}}(z)),\ \mathrm{Re}(z) \geq 0.$$

However, we must make a slight adjustment when computing the c.d.f F_{S_N} since the measure $\mu(\mathrm{d}x) := \mathbb{P}(S_N \in \mathrm{d}x)$ may have an atom at zero, and consequently F_{S_N} may be discontinuous there. We find that better numerical results are achieved if we remove this atom.

More specifically, let $\mu(\{0\}) = p_0 \in (0,\ 1)$. The measure $\mu_0(\mathrm{d}x) := \mu(\mathrm{d}x) - p_0\delta_0(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure. It is easy to see that $F_0(x) := \mu_0([0, x])$ has Laplace transform $\phi_0(z) := (\phi_{S_N}(z) - p_0)/z,\ \mathrm{Re}(z) \geq 0$. Therefore, we can use

$$F_{S_N}(x) = p_0 + \mathcal{L}^{-1}\left\{\phi_0(z)\right\}(x)$$

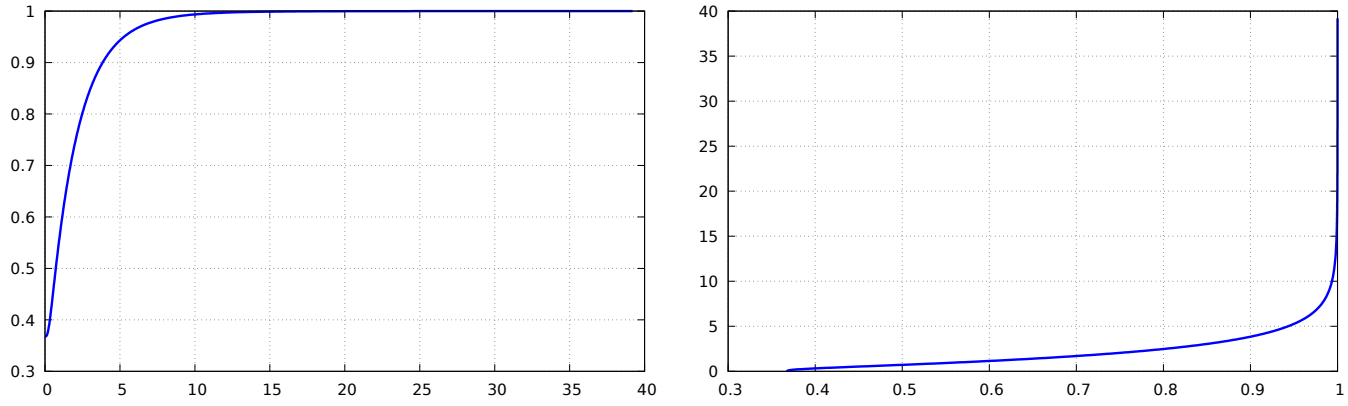
for the values of x near zero, and we revert to the methods in our preliminary examples otherwise.

Example 4. Let N be distributed Poisson with the unit rate parameter, and let $S_N = L_1 + L_2 + \cdots$, where $L_1,\ L_2, \ldots$ are mutually independent, independent of N, and such that $\sigma_1 = \sigma_2 = \cdots = 0.83$. For this special r.v. N, we have $p_0 = \exp(-1)$. The graph of the approximation $F_{\widetilde{S}_N}(x)$ is shown in Figure 7. Also shown therein is $\mathrm{VaR}_q[\widetilde{S}_N]$, which can be computed as the inverse of $F_{\widetilde{S}_N}(x)$ as before. Of course in this case we have $\mathrm{VaR_q[S_N]} = 0$ for $q \leq \exp(-1)$. Using these results, we also compute $\mathrm{CTE}_q[\widetilde{S}_N]$ and $\mathrm{mTV}_q[\widetilde{S}_N]$, which are depicted in Figure 8. This completes Example 4.

7 Computation time

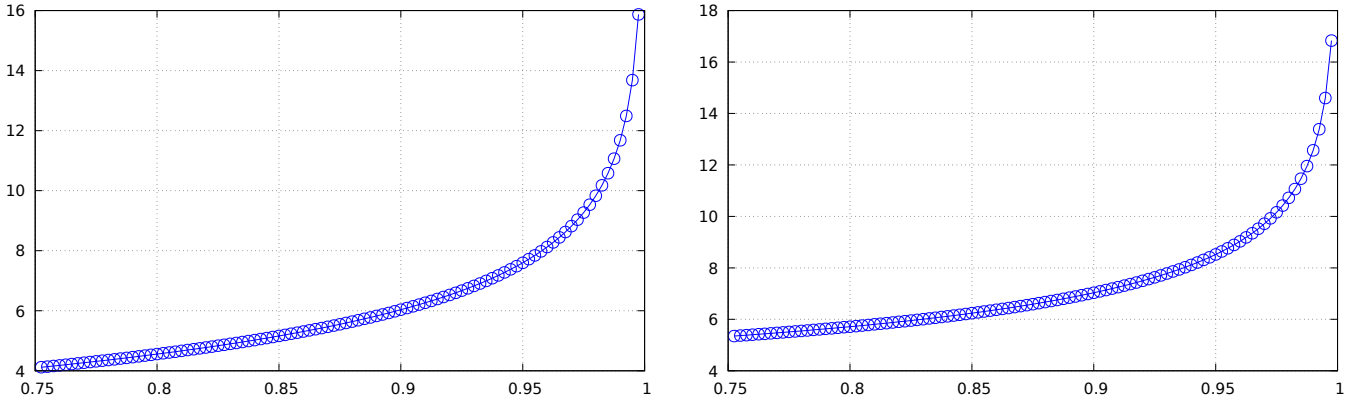
In this section we provide the computation time needed to compute the quantities of interest in the previous sections. These computation times are presented in Table 3.

All calculations were carried out on a desktop computer with 32GB of memory and an Intel i7-2600K 3.40GHz CPU. Times shown are measured in seconds. These represent the number of seconds required



(a) (b)

Figure 7: Degree $m = 20$ approximations (a) $F_{\widetilde{S}_N}(x)$ and (b) $\mathrm{VaR}_q[\widetilde{S}_N]$.



(a) (b)

Figure 8: Degree $m = 20$ approximations (a) $\mathrm{CTE}_q[\widetilde{S}_N]$ and (b) $\mathrm{mTV}_q[\widetilde{S}_N]$.

Approximation of the quantity	Pre-Comp. Integ.	W/O Pre-Comp. Integ.	10^6 MC	10^7 MC
$\text{VaR}_{0.5}[S_3]$	0	N/A	0	0
$\text{CTE}_{0.5}[S_3]$	0.56	7.41	0.35	3.46
$\text{mTV}_{0.5}[S_3]$	1.06	211.83	0.36	3.46
$\text{CTE}_{0.5}[L_1,\, S_3]$	0.92	6.38	0.36	3.53
$\text{mTCov}_{0.5}[L_1,\, S_3]$	3.25	43.67	0.35	3.45

Table 3: Computation times for various quantities of interest in the paper, for $S_3 = L_1 + L_2 + L_3$, where $L_1 \sim LN(0.81),\ L_2 \sim LN(0.83)$, and $L_3 \sim LN(0.85)$.

to compute one value of the approximation of the quantity listed in the left-most column by the method listed in the top-most row. None of the times include the time required to compute the coefficients of the approximation – these were pre-computed and stored on hard drive (the coefficients are available for download at www.math.yorku.ca/~akuznets/math.html for a range of values of $\sigma \in (0, 3)$ and for $m \in \{10, 20, 30, 40\}$). The column "Pre-Comp. Integ." shows the timings for when we have pre-computed and stored the integrands for the various Laplace inversions we have to do. Pre-computing these integrands takes a rather long time (due to the fact that we use a large number of discretization points in the inverse Laplace transform – see Appendix A), and if we compute multiple quantities of interest - and this is the case herein - it is more efficient to pre-compute these integrands and store them rather then compute them every time. For a comparison, the column "W/O Pre-Comp. Inteq." shows the computation time that includes the time used to build the integrands. The two columns on the right show the time taken by the Monte Carlo method.

8 Concluding discussion

Convolutions of log-normally distributed r.v.'s play a prominent role in actuarial science, and well-beyond it. Nevertheless, the existing methods, which as a rule hinge on either one of (i) the moment matching technique, (ii) series expansion of the Laplace transform, and (iii) asymptotic analysis, may deliver inaccurate results. Rather unfortunately, the just-mentioned inaccuracies exacerbate when, e.g., the intermediate tail (also tail) risk is of interest, and this is precisely the phenomenon that concerns the modern insurance regulation the most. In this paper we have proposed a hybrid approach to resolving

the problem. More specifically, we have shown that it is possible to approximate the distribution of the sum of independent and log-normally distributed r.v.'s with the help of the distribution of certain m-fold gamma convolutions in such a way that, for $m \to +\infty$, the approximating distribution is equal to the distribution of the desired sum. We then have utilized the class of Padé approximations to find the parameters of the approximating distribution.

Remarkably, the algorithm that arises from our method is fairly fast, accurate, and, last but not least, very versatile. We have discussed the two former advantages earlier in the paper. For the latter advantage, we note that our approach can be used to approximate the c.d.f. of the aggregate r.v. $S_n = X_1 + \cdots + X_n$, where the summand r.v.'s $X_j,\ j = 1, \ldots, n$ can have any - not necessarily common - distributions in the class of GGC's (Miles et al., 2019, for a non-technical follow-up paper with examples)

Acknowledgements

We thank the anonymous reviewers for their careful reading of the paper, thoughtful comments and questions. Also, we are grateful to the Casualty Actuarial Society for an award under the umbrella of the 2017 Individual Grants Competition that has enabled us to concentrate our time and energy on this project. Edward Furman and Alexey Kuznetsov acknowledge the continuous financial support of their research by the Natural Sciences and Engineering Research Council of Canada. Daniel Hackmann was supported by the Austrian Science Fund (FWF) under the project F5508-N26, which is part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

References

Allen, G. D., Chui, C. K., Madych, W. R., Narcowich, F. J., and Smith, P. W. (1975). Padé approximation of Stieltjes series. *Journal of Approximation Theory*, 14:302–316.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Asmussen, S. (2018). Conditional Monte-Carlo for sums, with applications to insurance and finance. *Annals of Actuarial Science*, 12(2):455–478.

Asmussen, S., Jensen, J. L., and Rojas-Nandayapa, L. (2011). A literature review on log-normal sums. Technical report, University of Queensland.

Asmussen, S., Jensen, J. L, and Rojas-Nandayapa, L (2016a). On the Laplace transform of the lognormal distribution. *Methodology and computing in applied probability*, 18:441–458.

Asmussen, S., Jensen, J. L, and Rojas-Nandayapa, L (2016b). Exponential family techniques for the lognormal left tail. *Scandinavian Journal of Statistics*, 43:774–787.

Asmussen, S. and Rojas-Nandayapa, L. (2008). Asymptotics of sums of lognormal random variables with Gaussian copula. *Statistics and Probability Letters*, 78(16):2709–2714.

Avram, F., Chedom, D. F., and Horváth, A. (2011). On moments based Padé approximations of ruin probabilities. *Journal of Computational and Applied Mathematics*, 235:3215–3228.

Baker, G. A. and Graves-Morris, P. (1996). *Padé Approximants*. Cambridge University Press, Cambridge-New-York, second edition.

Beaulieu, N. C., McLane, P. J., and Abu-Dayya, A. A. (1995). Estimating the distribution of a sum of independent lognormal random variables. *IEEE Transactions on Communications*, 43(12):2869–2873.

Bickerstaff, D. R. (1972). Automobile collision deductibles and repair cost groups: the log-normal model. *Proceedings of the Casualty Actuarial Society*, LIX:68.

Bondesson, L. (1992). *Generalized Gamma Convolutions and Related Classes of Distributions and Densities*. Springer-Verlag, New-York, lecture notes in statistics edition.

Bondesson, L. (2002). On the Levy measure of the lognormal and the logcauchy distributions. *Methodology and Computing in Applied Probability*, (4):243 – 256.

Botev, Z. I., Salomone, R., and Mackinlay, Daniel. (2018). Accurate computation of the distribution of sums of dependent log-normals with applications to the Black-Scholes model. Technical report. Available on arXiv: :1705.03196, accessed on January 08, 2019.

Chen, S., Nie, H. and Ayers-Glassey, B. (2008). Lognormal sum approximation with a variant of Type IV Pearson distribution. *IEEE Communications Letters*, 12(9):630–632.

Denault, M. (2001). Coherent allocation of risk capital. *Journal of risk*, (514):1–34.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2006). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. John Wiley & Sons, The Atrium, Southern Gate, Chichester.

Denuit, M., Dhaene, J., and Ribas, C. (2001). Does positive dependence between individual risks increase stop-loss premiums? *Insurance: Mathematics and Economics*, 28(3):305–308.

Dhaene, J., Henrard, L., Landsman, Z., Vandendorpe, A., and Vanduffel, S. (2008). Some results on the CTE-based capital allocation rule. *Insurance: Mathematics and Economics*, 42(2):855–863.

Dhaene, J., Tsanakas, A., Valdez, E. A., and Vanduffel, S. (2012). Optimal capital allocation principles. *Journal of Risk and Insurance*, 79(1):1–28.

Dropkin, L. B. (1964). Size of loss distributions in workmens compensation insurance. *Proceedings of the Casualty Actuarial Society*, LI:198.

Dufresne, D. (2008). Sums of log-normals. In *Actuarial Research Conference Proceedings*.

EIOPA-14-322 (2014). The underlying assumptions in the standard formula for the Solvency Capital Requirement calculation. Technical report, European Insurance and Occupational Pensions Authority.

Fenton, L. F. (1960). The sum of log-normal probability distributions in scatter transmission systems. *IRE Transactions on Communications Systems*, 8(1):57–67.

Filon, L. N. (1928). On a quadrature formula for trigonometric integrals. *Proceedings of the Royal Society of Edinburgh*, 49:38–47.

Fosdick, L. D. (1968). A special case of the Filon quadrature formula. *Mathematics of Computation*, 22:77–81.

Furman, E. and Landsman, Z. (2005). Risk capital decomposition for a multivariate dependent gamma portfolio. *Insurance: Mathematics and Economics*, 37(3):635–649.

Furman, E. and Landsman, Z. (2006). Tail variance premium with applications for elliptical portfolio of risks. *ASTIN Bulletin: The Journal of the International Actuarial Association*, 36(2):433–462.

Furman, E. and Zitikis, R. (2008a). Weighted risk capital allocations. *Insurance: Mathematics and Economics*, 43(2):263–269.

Furman, E. and Zitikis, R. (2008b). Weighted premium calculation principles. *Insurance: Mathematics and Economics*, 42(1):459–465.

Gulisashvili, A. and Tankov, P. (2016). Tail behaviour of sums and differences of log-normal random variables. *Bernoulli*, 22(1):444–493.

Holgate, P. (1989). The log-normal characteristic function. *Communications in Statistics - Theory and Methods*, 18(12):4539–4548.

Hürlimann, W. (2001). Analytical evaluation of economic risk capital for portfolios of gamma risks. *ASTIN Bulletin: The Journal of the International Actuarial Association*, 31(1):107–122.

Hürlimann, W. (2003). Conditional Value-at-Risk bounds for compound Poison risks and a normal approximation. *Journal of Applied Mathematics*, 3:141–153.

Jiang, C.-F., Peng, H.-Y., and Yang, Y.-K. (2016). Tail variance of portfolio under generalized Laplace distribution. *Applied Mathematics and Computation*, 282:187–203.

Kaas, R., Goovaerts, M., Dhaene, J., and Denuit, M. (2008). *Modern Actuarial Risk Theory Using R*. Springer-Verlag, Berlin, Heidelberg, second edition.

Kleiber, C. and Kotz, S. (2003). *Statistical Size Distributions in Economics and Actuarial Sciences*. Wiley & Sons, Hoboken, New Jersy.

Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2012). *Loss Models: From Data to Decisions*. Wiley & Sons, Hoboken, New Jersy, third edition.

Laub, P. J., Asmussen, S., Jensen, J. L., and Rojas-Nandayapa, L. (2016). Approximating the Laplace transform of the sum of dependent lognormals. *Advances in Applied Probability*, 48(A):203–215.

Leipnik, R. B. (1991). On log-normal random variables: 1 - the characteristic function. *Journal of Australian Mathematical Society Series B*, 32:327–347.

Limpert, E., Stahel, W. A., and Abbt, M. (2001). Log-normal distributions across the sciences: Keys and clues. *BioScience*, 51(5):341–352.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press, New-York.

Mikosch, T. (2009). *Non-Life Insurance Mathematics: An Introduction with Poisson Process*. Springer-Verlag, Berlin, Heidelberg.

Miles, J. (2018). The Laplace transform of the lognormal distribution. *arXiv: 1803.05878*, preprint.

Miles, J., Furman, E., and Kuznetsov, A. (2019). Risk aggregation: A general approach via the class of Generalized Gamma Convolutions. *Variance*, in press.

Milevsky, M. A. and Posner, S. E. (1998). Asian options, the sum of lognormals, and the reciprocal gamma distribution. *Journal of Financial & Quantitative Analysis*, 33(3):409–421.

O'Neill, B. and Wells, W. T. (1972). Some recent results in lognormal parameter estimation using grouped and ungrouped data. *Journal of the American Statistical Association*, 67(337):76–80.

Patil, G. P. and Rao, C. R. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics*, 34(2):179–189.

Ramsay, C. M. (2009). The distribution of compound sums of Pareto distributed losses. *Scandinavian Actuarial Journal*, 2009(September 2014):27–37.

Seal, H. L. (1980). Survival probabilities based on Pareto claim distributions. *ASTIN Bulletin: The Journal of the International Actuarial Association*, (11):61 – 74.

Sprenkle, C. M. (1964). Warrant prices as indicator of expectation. *Yale Economic Essays*, 1:412–474.

Takahasi, H. and Mori, M. (1974). Double exponential formulas for numerical integration. *Publ. RIMS Kyoto Univ*, 9:721–741.

Thorin, O. (1977). On the infinite divisibility of the lognormal distribution. *Scandinavian Actuarial Journal*, 1977(3):121–148.

Usábel, M. (2001). On the infinite divisibility of the lognormal distribution. *ASTIN Bulletin: The Journal of the International Actuarial Association*, 31(1):59–79.

Vanduffel, S., Chen, X., Dhaene, J., Goovaerts, M., Henrard, L., and Kaas, R. (2008). Optimal approximations for risk measures of sums of lognormals based on conditional expectations. *Journal of Computational and Applied Mathematics*, 221(1):202–218.

Venter, G. G. (2004). Capital allocation survey with commentary. *North American Actuarial Journal*, 8(2):96 – 107.

Wang, S. S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin: the Journal of the International Actuarial Association*, 26(26):71 – 92.

Zhang, K. Q. T. and Song, S. H. (2008). A systematic procedure for accurately approximating lognormal-sum distributions. *IEEE Transactions on Vehicular Technology*, 57(1):663–666.

Appendix A Numerical inversion of Laplace transform

In this section we remind the reader some results from the theory of Laplace transform and discuss how to compute the inverse Laplace transform efficiently and accurately. Let f be a p.d.f. and define by ϕ the corresponding Laplace transform

$$\phi(z) = \mathcal{L}f(z) = \int_0^\infty e^{-zx} f(x)\mathrm{d}x, \quad \mathrm{Re}(z) \geq 0.$$

The following results are well-known:

(i) If f is sufficiently smooth, then

$$f(x) = \mathcal{L}^{-1}\{\phi(z)\}(x) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{zx}\phi(z)\mathrm{d}z, \tag{44}$$

where c is an arbitrary positive number.

(ii) Let F be the c.d.f. corresponding to f and $\bar{F} := 1 - F$. Then $\mathcal{L}F(z) = \phi(z)/z$ and $\mathcal{L}\bar{F}(z) = (1 - \phi(z))/z$.

Let us now discuss how to compute the inverse Laplace transform efficiently. Assume that the function ϕ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and it converges to zero uniformly as $z \to \infty$ in this domain. Note that these conditions, while rather restrictive in general, are always satisfied in examples used in the present paper. We begin by writing the inverse Laplace transform in an equivalent form

$$f(x) = \mathrm{Im}\left(\frac{1}{\pi}\int_c^{c+\mathrm{i}\infty} e^{zx}\phi(z)\mathrm{d}z\right), \quad x \geq 0, \tag{45}$$

which follows from the fact $\overline{\phi(z)} = \phi(\overline{z})$. Now we choose a in the second quadrant (that is, $\arg(a) \in (\pi/2, \pi)$), rotate the contour of integration and change the variable of integration $z = c + au$ and obtain the following integral representation

$$f(x) = \mathrm{Im}\left(\frac{ae^{cx}}{\pi}\int_0^{\infty} e^{u\,\mathrm{Re}(a)x}e^{\mathrm{i}u\,\mathrm{Im}(a)x}\phi(c + au)\mathrm{d}u\right), \quad x \geq 0. \tag{46}$$

The above integral is more convenient to work with, compared to (45), for the following reason: the integrand decays exponentially fast in (46) as $\mathrm{Re}(a) < 0$.

In the computations in Sections 4, 5 and 6, we typically choose $-0.5 \leq \mathrm{Re}(a) \leq -0.1$, $\mathrm{Im}(a) = 1$ and $0.25 \leq c \leq 5$. To compute the oscillatory integral (46), we use the Filon's quadrature (Filon, 1928; Fosdick, 1968). This entails approximating $\phi(c + au)$ locally by a polynomial of degree two (using three discretization points), and integrating the result against the exponential function. In total, we typically truncate the integral in (46) at a point in the interval $[250, 500]$ and use between 500,001 and 1,000,001 unevenly spaced discretization points to evaluate the resulting integral.